LA-24

C. 1

For Reference

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LA REPORT 24

LECTURE SERIES ON NUCLEAR PHYSICS

~~Jane D. Hall.~~

CORRECTIONS

p. 241 - Equation 48 should read

$$\sigma(\alpha,\beta) = \pi \lambda\!\!\!^{-2} \left| \sum_n \frac{\gamma^n_{a\alpha l}\, \gamma^n_{b\beta l'}}{\epsilon - \epsilon_n + i\Gamma^n/2} e^{i\phi_{\alpha\beta}} + f(\epsilon) \right|^2$$

p. 254 - Missing only because of an error in numbering and not because of censorship.

p. 276 - Equation at top of page should read

$$\sigma(n,\gamma) = \xi\left(440/\sqrt{\epsilon}\right) \quad \text{for} \quad \epsilon < 10^4$$

LA 24 (1)

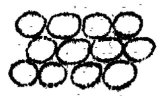
September 14, 1943.

LECTURE SERIES ON NUCLEAR PHYSICS

FIRST SERIES: TERMINOLOGY LECTURER: E. M. MCMILLAN

LECTURE I: THE ATOM AND THE NUCLEUS

We consider a piece of matter, for instance copper. If we magnify it, we first see that it consists of crystallites. They in turn are regularly arranged arrays of atoms. In the special case of copper, the geometrical arrangement is the same as a close packed arrangement of spheres, the atoms in one plane making the following pattern:



At the center of each atom there is a nucleus. Of course, the picture is not really like the one drawn. The electrons fill the whole space, there are no boundaries, the atoms actually overlap, their limits are fuzzy. In a metal like copper it is not even clear which electron belongs to which atom. In an insulator, for instance the noble gas Argon, each atom holds on to its own electrons and each electron belongs to a definite atom. In a chemical compound, for instance carbon dioxide, the electrons are shared by the atoms belonging to one molecule, but each molecule has its own electrons. All the chemical properties of the atoms are determined by the outer electrons.

The question arises what effect the nucleus has on the behavior of the atom. The effect of the nucleus is due to its <u>electric charge.</u> This charge is an integral multiple of the positive charge which is of the same magnitude as the charge of the electron. Atoms must be neutral because a positive charge would result in the attraction of more electrons, whereas a negative charge would cause the atoms to lose electrons. Each atom has enough (negatively charged) electrons to compensate the nuclear charge. This number of electrons is also called the atomic number and determines the chemical properties of the atom.

Another significant property of the atomic nucleus is the nucleus mass. This mass is much greater than the mass of the electrons. Even the lightest nucleus, namely of hydrogen, is 1840 times heavier than that of an electron. Thus the mass of matter is essentially concentrated in the nuclei. The mass of the nucleus depends in first approximation on the number of neutrons and protons within the nucleus. The neutrons and protons have approximately equal mass but this equality does not hold as exactly as the equality of positive and negative charges within an atom. The protons carry one unit of positive charge. The neutrons are uncharged. The charge of the nucleus, that is the atomic number, is equal to the number of protons in the nucleus. The total weight of the nucleus on the other hand, is roughly equal to the weight of a proton (or a neutron) multiplied by the total number of neutrons and protons within the nucleus. This number is called the mass number. Thus if two nuclei have the same charge, it does not mean that they have the same mass number because a given number of protons might be associated with a different number of neutrons in different nuclei. If two nuclei have the same number of protons they will belong to the same chemical element (atomic species) but if, at the same time they have a different number of neutrons they will be different isotopes. For instance hydrogen has different kinds of isotopes. Every hydrogen nucleus has one proton but it may have 0, 1 or 2 neutrons. There exist many more nuclear species than atomic species. In order to designate the nuclear species we write the atomic symbol, for instance H in the case of hydrogen. This already implies what the nuclear charge is and therefore shows how many protons are found in the nucleus. The mass number of the nucleus is shown by an upper index. The following table gives the number of neutrons and protons for the known hydrogen and helium isotopes:

	number of protons	number of neutrons
H^1	1	0
H^2	1	1
H^3	1	2
He^3	2	1
He^4	2	2
He^6	2	4

The approximate <u>size of an atom</u> is 10^{-8} cm. Different atomic species have different size but the total variation in size is not more than a factor of ten times. One may visualize the size of an atom by expanding our scale of measurements in such a way that 1 cm. will appear as 1,000 miles. Then an atom will be about 1 cm. in radius. The size of the atomic nucleus is about 10^{-13} cm., that is, about 10^5 times smaller than the atom. The distance between the sun and earth is about 100 times the radius of the sun. Thus the nucleus is relatively much smaller from the point of view of its outer electrons than the sun as viewed from the earth.

It was stated that the lightest nucleus is about 2,000 times heavier than an electron. We know that in 1 gm. of hydrogen there are 6×10^{23} atoms, therefore the mass of one hydrogen nucleus is $1/6 \times 10^{-23}$ grams.

Before turning to the dimension of the electron, we shall discuss the peculiar difficulty which arises in connection with visualizing the electrons within the atoms. It is often stated that an atom is like a solar system. The atom of iron for instance has 26 electrons. Can we measure these as in motion in approximately elliptical orbits around the nucleus? If we should take a microscope of very high power and resolution and take a photograph of the inside of an atom at a given instance, one would find a disorderly arrangement of electrons within the atom. One might then try to take another photograph a very short time later in order to find the motion of the electrons within their orbits. If such a photograph is taken no relation is discovered between the positions in the two consecutive pictures. Actually if the resolving power of the microscope would have been infinity, the second picture would have been blank. The reason for this situation is that the light which is used in taking the pictures carries momentum and one cannot therefore take a photograph without giving a kick to the electrons. If good resolving power is wanted, you must use light of very short wave length and such light carries particularly high momentum. One might try to compromise in following the orbit of the electrons by using a lower resolving power and hoping that in this way the velocity of the electrons will not be too much disturbed in taking a photograph but the quantitative situation is such that the necessary uncertainties in position and momentum are just big enough to make it impossible to define orbits within atoms.

This conclusion is a consequence of quantum mechanics. Quantum mechanics is the mechanics that applies to all bodies, for instance to a tennis ball, but in the case of a large body like a tennis ball the laws of quantum mechanics approach the laws of classical mechanics so closely that the two become practically undistinguishable. The word quantum means amount and occurs in the designation quantum mechanics because many of the characteristic laws of quantum mechanics are associated with the fact that certain quantities appear in amounts that are multiples of a given characteristic amount. The most basic of these quantities is the so-called "quantum of action" which is designated by h. This quantity has the dimension of a distance times the momentum. h is the measure of the limitation on measurement. If the position of a particle (i.e. its distance from a reference system) is well known, then little can be known about its momentum and vice versa. If the momentum is known the position must be unknown. The product of the uncertainties in momentum and position is of the magnitude h. For a big mass this uncertainty does not mean much. This is so because the momentum is a product of mass and velocity and if the mass is big, the uncertainty in the velocity becomes correspondingly small, but for the lightest particles, the electrons, the uncertainty of momentum causes such an uncertainty in velocity that to define anything like an orbit for an electron becomes impossible. The only statements that we can make about the position of electrons in an atom are of a statistical nature.

One might for instance take many instantaneous photographs of a certain kind of atom in the way described above, one might then superimpose these photographs and we then find that the electron position will distribute itself more in certain regions (near the nucleus) than in other regions. Such a picture gives an idea of the probable density of the electrons inside the atom, i.e. the probability with which the electrons can be found in the various volume elements inside the atom.

The description just given might induce one to say that an electron is as big as the whole atom. However, in an individual measurement the position of an electron may be more sharply defined and in fact electrons do have their own radii which are much smaller than the atomic radius. The concept of the electron radius can be understood by considering the electric field of an electron. This electric field is according to the Coulomb Law inversely proportional to the square of the distance from the electron. If the electron were a point, the field near the electron would be infinite. It is not believed that such infinite fields can exist and it is assumed that at some given distance from the electron, the Coulomb Law ceases to hold. This distance has been estimated as about 10^{-13} cm. That is the electron radius and the nuclear radii are of comparable size. Of course, the position of the nuclei is not quite so hard to define as the position of electrons due to the fact that the nuclei have much bigger mass and according to that reasoning uncertainties in their momenta, will not cause very great uncertainties in their velocities and will not wash out their orbits to the same extent as those of the electrons.

In some of the applications of quantum mechanics which we shall encounter later, the wave lengths associated with particles will play a fundamental role. One length associated with a given particle is h/mv (h = quantum of action; m = mass of particle; v = velocity of particle; mv = momentum of particle). This length is actually equal to the uncertainty of position associated with an uncertainty in momentum of the magnitude mv. De Broglie was the first to postulate that a wave process is associated with every particle and it is shown that the wave length is h/mv. This association between particles and wave processes is surprising but it has been actually demonstrated by interference experiments in which electrons or light atoms (for instance hydrogen) have been reflected from crystal surfaces. The laws of this reflection process are of the same kind as the laws of X-ray reflection and have been described in terms of wave interference. It is apparent from the above statements that the de Broglie wave length is closely related to the uncertainty of electron position and also to the fuzziness of atomic boundaries.

September 16, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

FIRST SERIES: TERMINOLOGY LECTURER: E.M. MCMILLAN

LECTURE II: INTERACTION OF LIGHT WITH MATTER

The list of simple particles mentioned in the last lecture comprised electrons, protons and neutrons. We may add to this list light. It is known that light consists of electromagnetic waves, but if, as has been mentioned, the wave process is associated with electrons, neutrons or protons, which we ordinarily consider as particles, then it will not be too surprising to find that light, which ordinarily can be thought of as a wave process, has certain particle properties. When light behaves like a particle, we talk about a light quantum.

There is an important difference between light and proper particles, for instance the electrons. The electrons have a strong tendency to persist. Light quanta do not. They get absorbed and emitted. It is not easy to count them and for this reason it is impracticable to consider them as constituents of matter. On the other hand, whenever light does get absorbed or emitted, a chunk of energy of a definite size disappears or is produced in each such process as though light did consist of particles carrying a certain amount of energy. This energy of a light quantum is $h\nu$, where ν is the frequency of the light considered and h is the same quantum constant which appears in the de Broglie relation connecting the momentum of a particle with the wave length associated to it.

If the light quantum possesses a definite energy $h\nu$, one will also expect it to carry a momentum. According to the theory of relativity this momentum is $h\nu/c$, where c is the light velocity. It will be noticed that the ratio of energy to momentum is not the same as in Newtonian mechanics. If we write for the kinetic energy $mv^2/2$ (m is mass of particle; v, its velocity) and for the momentum mv, then the ratio of the 2 quantities is $v/2$, that is one half of the velocity of the particle rather than the velocity itself. But Newtonian mechanics applies only as long as the velocity v is small compared to the velocity of light c. The value $h\nu/c$ of the momentum of a light quantum can be easily seen to agree with the de Broglie relation. In fact, c/ν is equal to λ the wave length of light and

$$h/\lambda = \text{momentum}$$

is the same equation which also holds for electrons, neutrons and protons.

In discussing light quanta, it was necessary to introduce relativity. While this branch of physics, as well as quantum mechanics mentioned last time, have an elaborate mathematical foundation, we need to be concerned here only with a few simple consequences of these theories. Nuclear physics too will be treated here on a similarly simple descriptive plane, in which the elementary particles, electrons, protons and neutrons, are introduced without describing in detail the evidence that they are elementary particles and without discussing the difficulties to which this concept leads. We may add here to the list of the proper particles, the positron, whose existence was predicted by the combined application of quantum mechanics and relativity. The positron has the same properties as an electron, except that it carries a positive rather than a negative charge. The only reason why its discovery occurred much later than that of the electron is that the positron is not stable in the presence of electrons. The positron is attracted by an electron to which it happens to come close. The charges of the two particles compensate each other and their energy is radiated out in the form of light. Thus an electron-positron pair is annihilated. It is possible to produce positrons but in the presence of electrons which are found in all ordinary matter they can exist only a very short time. Actually the production of positrons is the reverse process of the annihilation; in each production process, an electron-positron pair appears, therefore, when a production-annihilation cycle is ended, we are left with the original number of electrons.

In addition to all these particles, we shall encounter two more half-known kinds of particles, the mesotrons and the neutrinos. The mesotrons are known experimentally. They are created in special processes observed in cosmic-ray physics but they have not been observed up to now in ordinary nuclear reactions Like the positrons they disappear shortly after their creation. They are of importance in the theory of nuclear forces which at present is widely accepted. The neutrinos have not been observed. They may be considered at present as a pure invention introduced to satisfy the conservation laws of energy of momentum in certain well known nuclear reactions in which these conservation laws are apparently violated.

One important consequence of the theory of relativity is the connection between mass and energy. A moving particle is heavier than the same particle at rest. The difference in mass is proportional to the kinetic energy. Potential energy has a similar influence on mass. The proportion constant between energy and mass is the square of the light velocity

$$\text{Energy} = mc^2$$

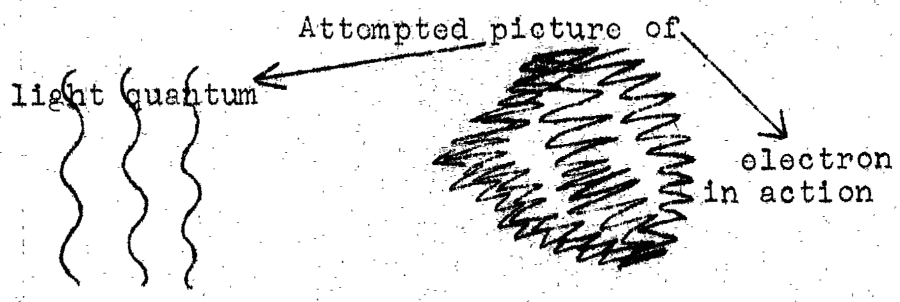
This relation is difficult to demonstrate under ordinary conditions in which the kinetic or potential energy per gram of material is rather small. In these cases, due to the high value of the light velocity, the energy changes due to the kinetic or potential energy are very small compared to the original energy of the masses involved, but in nuclear physics much greater energies are associated with a given mass, and here the changes in mass become detectable. We may consider the energy relations in a nuclear reaction in which two Deuterons (H^2 nuclei) unite to form α-particle (He^4)*. The reaction has actually not been observed. Its occurrence is very improbable, because conservation of momentum prevents the α-particle formed from carrying away the energy released by the reaction as kinetic energy. Therefore, the energy released must be omitted in the form of light and all reactions involving omission or absorption of light are improbable as compared to the reactions in which only heavy particles, neutrons, protons or other nuclei, are involved. It may be of interest that a collision of

* The lightest nuclei, namely proton, Deuteron and α-particle, have these special names in addition to their more systemic designation giving the name of the element and the mass number.

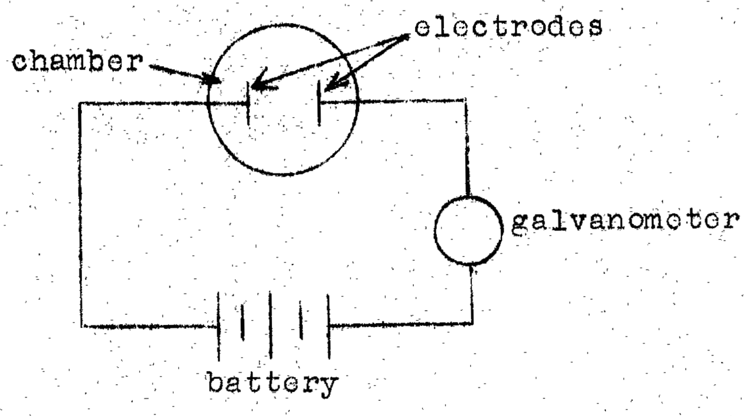
two Deuterons actually gives $H^3 + H^1$ or $He^3 + n$ (n stands for a neutron) with about equal probability. These reactions are much more probable than the reaction mentioned above.

Theoretically, $2H^2 \longrightarrow He^4 +$ light. This reaction can be used as a very simple example for the energy-mass equivalence. The mass of H^2 is 2.015 mass units while that of He^4 is 4.004 mass units. Therefore, the product nucleus He^4 is .026 mass units lighter than the two original H^2 nuclei. This difference of mass corresponds to a difference of energy and since in the reaction the nuclear mass and energy have decreased, the difference in energy becomes available (in the present case in the form of light) as the energy of the reaction. The quantitative connection between energy and mass works out in such a way that .001 mass unit or a millimass-unit corresponds approximately to 1 Million electron Volts (more accurately .935 MeV). The electron Volt (eV) in turn is an energy unit 1.6×10^{-12} ergs. It is the energy an electron acquires if it falls through a potential difference of 1 Volt. The relation between mass and energy is frequently used to find the exact value of a nuclear mass. This can be done if in a nuclear reaction the mass of all but one participants are known and if the energy of the reaction is measured.

In practical nuclear physics interaction of light quanta with matter plays an important role. One way of such interaction is <u>photo ionization.</u> In the figure



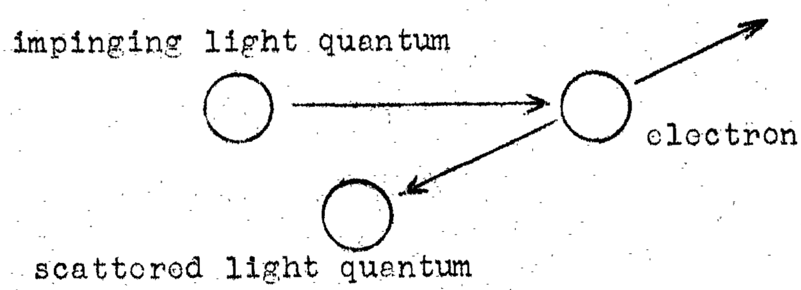
a light quantum impinging upon an atom is shown. A certain energy, say 10 eV, is needed to rip off the electron from the atom. If the $h\nu$ of the light quantum is smaller than this energy, the electron can not leave the atom. But if $h\nu$ exceeds 10 eV, the quantum may be absorbed; the electron is ripped off in the process and carries away the excess over the 10 eV in the form of kinetic energy. The remainder of the atom is no longer neutral but carries a positive charge. It is then called a positive ion. The corresponding negative ion, namely the electron itself, is also formed. Hence the name photo ionization, or the production of ions by light. All actual observation in nuclear physics depends on ionization. A typical arrangement for such observations is sketched in the following figure.



The air in the chamber is ordinarily an insulator and so no current can be sent by the battery through the galvanometer, but if ionization occurs in the chamber, the ions now moving up to the electrodes will carry a certain small amount of current which can be read in the galvanometer. Thus by the current one can measure the ionization.

Ionization can be produced by charged particles as well as by light. Actually the ionization by fast moving electrons, protons, Deuterons and α-particles, is a practical basis of experimental nuclear physics. These particles moving past atoms exert an electric field and if their velocity is high enough, produce ionization. Such particles may enter a chamber like the one shown in the above figure through a thin foil They are stopped by a thicker foil. One of the simple basic experiments of nuclear physics is to find out what is the foil thickness that will stop a certain kind of particle carrying a given energy.

There are other interactions between light and matter. One of them is the Compton effect. This is the scattering of a light quantum on a free electron, that is, one not bound in an atom. If the electron is not free but its binding energy is negligible compared to the energy of the light quantum, we may still talk of a Compton effect. In this effect, the behavior of light can be satisfactorily described by a particle picture and we therefore draw in the following figure the light quantum as a particle.



The momentum and energy imparted to the electron by the light quantum may be calculated by laws of conservation of energy and momentum in relativistic mechanics. The shorter arrow on the scattered light quantum indicated that the light quantum has given part of its energy and momentum to the electron. This is true if the electron was originally at rest, which in practice may be assumed almost without exception. The effect was discovered by A. H. Compton. He observed that most of the X-ray scattered through different angles has suffered an energy loss which depended on the angle. The energy loss agrees quantitatively with the one derived from the energy-momentum conservation laws. The Compton effect is the principal means by which γ-rays (that is, high frequency light $h\nu$ of the order of 1 MeV) lose their energy. High energy recoil electrons produced in the Compton effect may be detected by their ionizing power.

There exists a third kind of interaction between light and matter, this is the pair production. In this process one light quantum disappears and an electron-positron pair is created. This effect can occur only for high energy light quanta, namely, the energy of the light quanta must supply at least the mass of the electron and the positron. It may in addition supply an arbitrary amount of kinetic energy to these particles. Quantitatively the light quantum must carry approximately at least 1 MeV (more accurately, at least 1.024 MeV). The process can not occur in empty space. It requires the presence of matter. The reason for this is not that the process requires any other raw material than the energy carried by the light quantum but pair production in empty space would violate the law of conservation of momentum. The process can happen in the neighborhood of a nucleus. The nucleus having a great mass can take up almost arbitrary amounts of momentum. It therefore acts like a platform in the neighborhood of which electric fields can act to produce a pair and transfer any extra momentum to the nucleus. The process occurs with greatest probability near to strongly charged nuclei in whose neighborhood high electrostatic fields exist. The highest energy light quanta formed in cosmic rays lose their energy almost completely by the pair production process. The higher energy γ -rays which occur in nuclear processes lose energy primarily by the Compton effect and are absorbed only to a small extent by pair production process. As the energy of the γ-rays gets lower, photo ionization plays an increasingly important role in the energy loss in addition to the Compton losses.

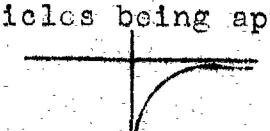
September 21, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

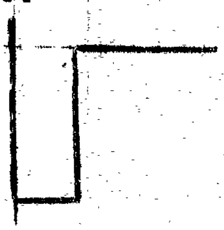
First Series: Terminology Lecturer: E. M. McMillan

LECTURE III: FORCES HOLDING THE NUCLEI TOGETHER

As the best known kind of force acting between particles, we might first consider electrostatic forces. We know that particles of like charge repel each other and particles of unlike charge attract each other, the force being inversely proportional to the square of the distance. It is a simplification to use the concept of potential energy instead of that of force. The potential energy is the amount of work needed to bring the particles from infinity where they do not interact to a definite distance from each other. If there is repulsion, work will need to be done and potential energy is positive. In case of attraction less than no work has to be done and the potential energy is negative. The potential energy is the integral of the force. For the inverse square law of electrostatics, the potential is proportional to one over the distance. The figure shows such an attractive potential, the distance between two particles being applied as abscissa and the potential as ordinate.

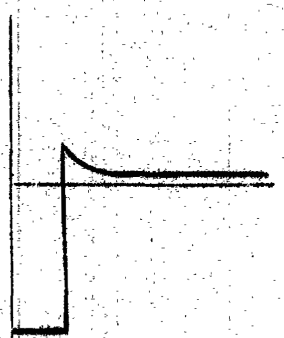


There is no electrostatic force between a neutron and a proton since the former does not carry a charge. Actually there appears to be no sizeable force acting between these two particles



unless they approach to a distance comparable to their radii which is about 10^{-13} cm. There a large attractive force suddenly appears as shown in the figure which shows the potential energy acting between these two particles. The detailed dependence of this potential on the distance is not known but the approximate shape appears to be that of a <u>potential well.</u> Its depth is considerable, namely several MeV.

A potential of this type acts not only between neutrons and protons, but also between two neutrons. Two protons will repel each other according to the laws of electrostatics at great dis-



tances, but at close distances the same kind of attraction appears as between neutrons and protons. The potential between two protons is illustrated in the graph. One can see in this graph the distinction between <u>long range forces</u> such as the electrostatic forces which act at a greater distance, and <u>short range forces</u> such as those due to the potential well whose effect is noticeable only if the particles are close. A common example that might make this difference clearer is the short range repulsive force occurring when rigid bodies are brought into contact and the more slowly varying force which appears when a spring is extended and which is somewhat analagous to a long range force.

If we now want to build up a nucleus from several neutrons and protons, it is necessary that at least the edges of these particles should touch so that the short range attractive forces may become operative. This requirement is not necessary in the case of the electron in the atoms, which are held together by the long range electrostatic attraction and in which therefore the average distance between particles is much greater than the radii of these particles. Another reason why the neutrons and protons constituting the nuclei can get so close together is the fact that heavy particles may more easily have smaller wave length. According to quantum mechanics, it is not possible to confine a particle within a small space without effectively giving it a similarly small wave length which according to the de Broglie relation means a high momentum. Heavy particles may possess such a high momentum without thereby obtaining an excessive kinetic energy which would surpass the nuclear binding forces and disrupt the nucleus. Since electrons confined to nuclei would have such a high kinetic energy we must not expect to encounter them as nuclear constituents. In other words, when an electron is attracted to a nucleus, the attraction will increase its kinetic energy and momentum and thereby will decrease its wave length, but when the electron has approached to within the distance of the nuclear radius its wave length is still not as short as the nuclear radius and therefore one can not think of it as being confined to the nucleus.*

There is no essential difference between the laws of nature operating in atoms and nuclei but in the atoms the light electrons do not as a rule come close enough so that the short range forces may become effective. It is not at all improbable that such short range forces act between electrons as between the heavier particles. Only cosmic ray electrons possessing very high

* Actually electrons may be squeezed into a smaller space than they usually occupy in atoms. This happens in dense stars. The energy which is necessary to give electrons high momentum associated with their more close confinement is derived in this case from gravitation.

energies may come close enough to other particles and in those cases it is reasonable to look for short range potentials.

The conclusion is that a nucleus tends to be a more or less closely packed assembly of neutrons and protons. Such an assembly is shown schematically. The crosses represent protons, the circles neutrons. The close arrangement is due to the short range attractions.

On the other hand, there is also a long range repulsion between the protons which tends to pull the nucleus apart. If we consider neutrons and protons mixed in a certain proportion and increase the total number of particles, we will expect that the long range repulsion will eventually prevail and destroy the nucleus. Indeed, the short range forces may be considered the same kind of force as cohesion. They will tend to give the nucleus a surface which is as small as possible, increasing the amount of matter and with it the nuclear radius (R). The stabilizing tendency of this force will be proportional to the surface and therefore to R^2. On the other hand, the total charge in such a nucleus is proportional to the volume, that is to R^3 and the disruptive force is the square of this charge divided by the square of the average distance between protons which again is R. Thus, the disruptive forces would increase as $\frac{R^3 \times R^3}{R^2} \sim R^4$ and they will become predominant with increasing R.

The structure as described above in a qualitative manner must not be taken too literally. There is a lot of kinetic energy in the nucleus and the particles must not be considered as localized. The protons, due to their mutual repulsion, have a tendency to be pushed out toward the surface, but also the number of neutrons and protons per unit volume tend not to differ too greatly. This will be particularly noticed in light nuclei most of which have an approximately equal number of neutrons and protons. The heaviest nuclei have almost twice as many neutrons as protons but there is reason to believe that a nucleus consisting of neutrons alone (or else protons alone) would not hold together at all. For a better understanding of the situation we shall have to explain the operation of the so-called <u>exclusion principle.</u> In order to apply this principle in practical cases, we also have to mention the <u>spin</u> of particles.

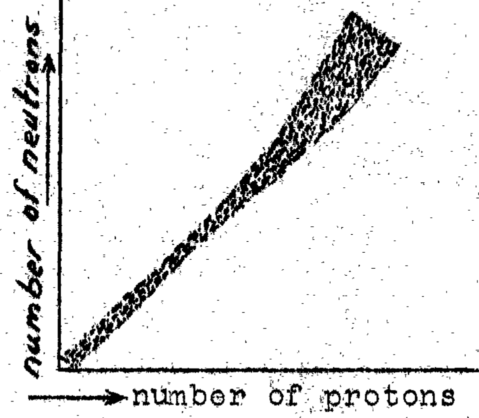
Neutrons, protons and also electrons carry an angular momentum or spin which in quantum units has the size of $\frac{1}{2}$. One may think of such a particle as rotating around an axis which might be oriented in space in different ways. For a complete description of such a particle, it is not sufficient to say where it is or in which orbit it moves, but it is also necessary to give the orientation of its rotational axis. It is a consequence of the application of quantum mechanics that for a particle with the intrinsic angular momentum $\frac{1}{2}$, the directional angular momentum may be described by simply stating that the spin points upward or downward. This quantization of direction is closely related to the quantization of angular momentum. It also should be remarked that, whereas the angular momentum of elementary particles can be observed, there is no direct way to investigate their angular velocity. There is, however, a further indication or rotation of such particles in their magnetic moment. Even neutrons, whose net charge is zero, possess such a magnetic moment.

The spin assumes its greatest importance in connection with the exclusion principle. This principle states that two like particles must not be in the same state. To show how this principle operates and how it is connected with the spin, let us first consider an imaginary spinless electron. One may classify the electron orbits around the helium nucleus by their energy. One electron will occupy the orbit of least energy. Then the second electron can not be assigned the same orbit because this would violate the exclusion principle. The second electron would have to move in a less strongly bound orbit and the helium atom would not have its particular stability. If, however, we now remember that electrons have a spin which is capable of pointing in two different directions, then it is possible to put two electrons differing in their spin directions into the lowest orbit of the helium atom, and a very stable arrangement is obtained. The third electron would have to agree in its spin direction with one of the two electrons and therefore the exclusion principle prevents it from being placed in the same orbit with the first two electrons. Actually, the exclusion principle requires that two electrons never be in the same state, that is, they must never be in the same orbit and at the same time have the same spin direction.

The same general consequence of the exclusion principle in atomic and molecular physics is the appearance of closed shells in atoms and a tendency of electrons to appear in pairs in stable molecules. There are actually only very few stable compounds with an odd number of electrons.

A similar tendency is seen in nuclei in which again the most stable ones possess an even number of neutrons and an even number of protons. The simplest nucleus of this kind is the particle, consisting of two neutrons and two protons. Due to the two possible spin orientations in neutrons and protons, the exclusion principle permits all particles in the He^4 nucleus to occupy the same (namely lowest) orbit. If now, one more neutron is added, it must go into a new orbit of higher energy. The resulting nucleus would be He^5 but the energy of that next orbit is so high that it is not held in at all by its connection with the other particles and He^5 does not exist. If two neutrons are added, leading to He^6, the attraction between the two additional neutrons stabilizes the nucleus and He^6 is actually known, but it is not very stable.

In general, the advantage of having an approximately equal number of neutrons and protons is easily seen. If too many particles of either kind were present, the exclusion principle would force them into new and higher orbits and one would "run out of space." An equal number of particles of the two kinds insures the greatest number of pairs of particles which can exercise their short range attraction. At the same time the number of times that the exclusion principle forces particles into new and higher orbits is reduced to a minimum. In heavier nuclei, however, it will be advantageous to have somewhat greater number of neutrons than protons in order to minimize the repulsive action between protons. One may represent nuclei in a graph by dots. The abscissa of each dot gives the number of protons in the nucleus, the ordinate gives the number of neutrons.



All known nuclei then fall in a rather narrow band which starts up at 45° corresponding to a roughly equal number of neutrons and protons. Later the band bends upward corresponding to the increased proportion in neutrons.

This discussion makes it more understandable why the number of particles in nuclei is limited. If too many protons were present in a nucleus, the electrostatic forces would disrupt the nucleus; if on the other hand, we start adding protons and increase the number of neutrons, then the exclusion principle will require the particles to go into such high orbits that the short range attractions are no longer sufficient to hold the nucleus together.

LA 24 (4)

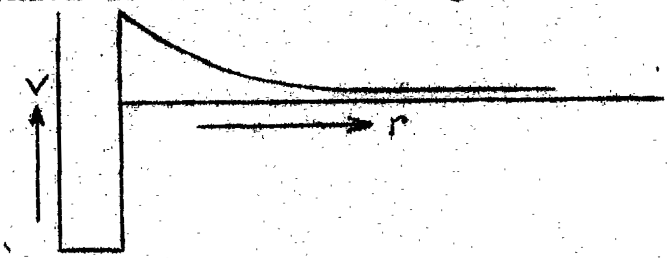
September 23, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

First Series: Terminology   Lecturer: E.M. McMillan

LECTURE IV: REACTIONS BETWEEN THE NUCLEI

In order to be able to discuss the reactions between the nuclei, we have to investigate the forces with which they act on each other. These forces may be obtained from their mutual potential energy, which is shown in the figure.

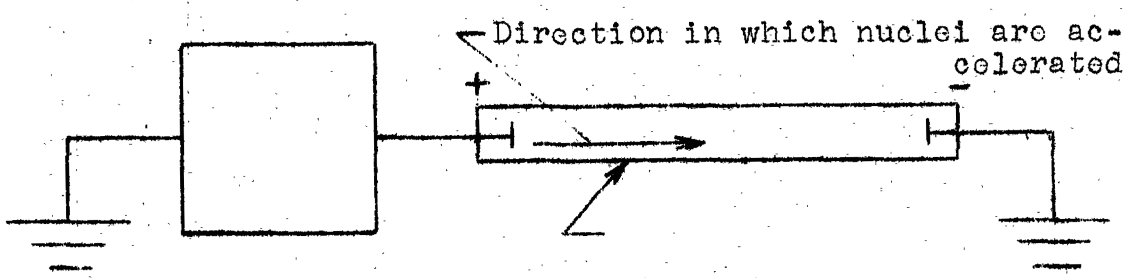


In this figure, the potential energy is plotted as a function of the distance of the nuclei r. At greater distances, the electrostatic repulsion between the nuclei is the only force acting. In this, the potential increases as the inverse first power of the distance. At smaller distances, the "stickiness" of the nuclei comes into play; that is, as soon as the nuclei touch, they attract each other. This results in a drop in potential energy which we have represented as a potential well. For not too heavy nuclei, this potential well is so deep as to over-compensate the electrostatic repulsion, and in this case the potential energy becomes negative, as has been actually shown in the figure.

The strength of repulsion between nuclei and therefore the height to which the potential barrier rises, depends on the product of nuclei charges. This product is smallest for H^2 atoms, but even in this case the potential barrier of a few hundred kV is to be expected. This potential value is of practical importance in the well-known reaction between two heavy H nuclei, which is to be discussed below. For all other pairs of nuclei, the potential barrier is even higher; that is, of the order of a few million volts, or for two heavy nuclei even considerably more.

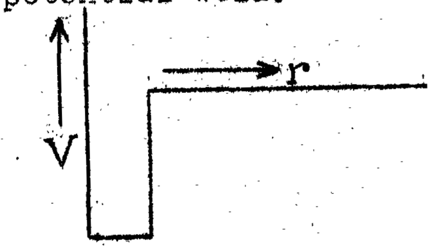
The strong repulsion between nuclei explains why nuclear reactions are uncommon under ordinary circumstances. Indeed, the kinetic energy of particles due to thermal motion at room temperature is about 1/40 of an electron volt, while energies encountered in chemical reactions are a few electron volts. These energies are far too little to allow nuclei to get close enough to each other so that they may touch and reaction may start. Nuclear reactions are known to proceed however in the interior of the sun, where temperatures of the order of many million degrees are found. In the laboratory, it is necessary to impart to the nuclei high kinetic energies, in order that they should get close enough to react.

In order to accelerate particles sufficiently, devices of two essentially different kinds have been used. The one is schematically shown in the figure:



Here an evacuated tube is shown in which the nuclei are accelerated from the positive electrode toward the negative electrode which is grounded. An essentially different method to impart high velocity to nuclei is the use of the cyclotron. In that device, the energy is given to the nuclei in a series of small pushes. The nuclei are held in circular paths by a strong magnetic field. The accelerating electric field is actually an oscillating field which exerts its accelerating action, always when the nuclei reach a certain part of the circle. While the particles get accelerated, they move in increasing circles and are finally ejected as fast particles.

The type of interaction between nuclei described in the beginning of this lecture is quite different from the interaction between nuclei and neutrons. This interaction potential is shown in the figure as a potential well.



There is of course no electrostatic repulsion between the neutrons and the nucleus. This is the reason why neutrons, though they are quite common particles in the nuclei, have remained unknown so long. A neutron can approach any nucleus freely, can react with that nucleus (for instance, it can be captured) and so it will stop being a free neutron*. Thus in the laboratory, a neutron may be seen just "on the fly". One can obtain them by bombarding a suitable nucleus by a suitable particle, for instance, a proton.

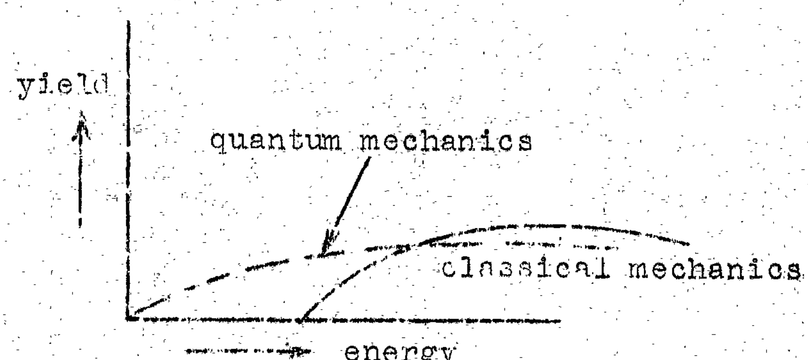
The particles which have been actually used as bombarding particles in nuclear reactions are either uncharged or lightly charged particles. A heavily charged particle requires much too high an energy to overcome the potential barrier, and approach another nucleus sufficiently close. In the following table, the first column gives the name of particles used in bombarding nuclei. If that particle is an atomic nucleus, its notation appears in brackets. The second column gives special symbols by which these frequently used particles are designated.

*There is another purely theoretical reason why free neutrons are difficult to observe. They are supposed to be unstable, and are expected to transform by radioactive action into protons emitting an electron at the same time, but a long time before this radioactive transformation could take place, a neutron is usually captured.

proton	(H^1)	p
deuteron	(H^2)	d
α-particle	(He^4)	α
neutron		n
radiation, or γ-rays		γ
electron		β

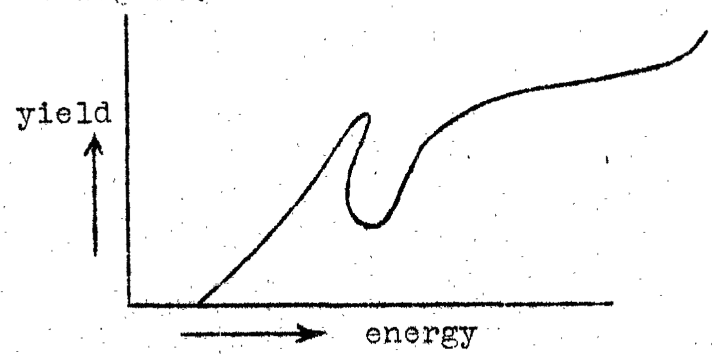
To this list, the nuclei H^3 and He^3 might be added, but they are practically unavailable and so they have not been used in our experiments of this kind. If one wants to use neutrons as bombarding particles, one has to derive them from nuclear reactions.

If the bombarding particle in a nuclear reaction is charged, one will expect no nuclear reaction to occur unless the bombarding particle has sufficient kinetic energy to overcome the potential barrier between the two particles. Thus, if the yield of such a reaction is plotted against the energy of the bombarding particle, one expects the yield to remain zero up to a given energy V and then rise in some manner. This behavior is shown in the figure by the full curve.



Actually one finds that this is not so, and that yields do not vanish below the point V, but rather have small but finite values in that region. This is due to the fact that in quantum mechanics it is not permissible to make the statement that the orbit of the bombarding particle extends exactly up to the maximum of the potential barrier. Due to the wave nature of particles, a certain fuzziness of the orbit is unavoidable and even though the energy does not suffice to carry the particle right to the top, there remains a small probability that the particle may "leak through" the barrier. But the fuzziness in quantum mechanics is only of a limited extent. Therefore, if the particle is given less and less energy and is therefore turned back according to classical mechanics at greater and greater distances from the top of the potential barrier, the probability that it can "leak through" the barrier decreases rapidly. The actual yield to be expected according to quantum mechanics is shown as a dotted line. At high energies, where the particle can get easily over the potential barrier, the yield may approach a constant value or may even decrease again.

The yield curve may show a sharp maximum. This is illustrated by the figure.



To understand this phenomenon, we must consider the bombarding and bombarded particle after they have touched and have formed a so-called compound nucleus. This compound nucleus has quantum levels and a sharp maximum occurs in the yield curve if the energy of the bombarding particle is just right for the formation of such a quantum level. Thus the maximum is due to an agreement between the energies of the particles before reaction and the energy state of the compound nucleus. Such an agreement is called resonance. The phenomena due to such resonance are analogous to those that are encountered in classical mechanics when the frequencies of two vibrating strings become equal to each other. The mathematical theory of these two situations are similar and the shape of the yield curve near a resonance maximum is essentially the same as certain resonance curves encountered in classical mechanics.

It may happen that in a reaction, the produced particles have greater intrinsic energy content than the initial particles. In this case, the reaction is possible only if this energy deficiency is supplied by the kinetic energy of the original particles. Then a minimum kinetic energy is required for the reaction to proceed, and the yield remains zero up to this energy. The energy at which the reaction first becomes possible is called the threshold of the reaction. Unlike the case of the potential barriers, this threshold is sharp, and below it the reaction cannot proceed even with a small probability.

As an example for a simple nuclear reaction, we may consider the reaction between two H^2 nuclei. There is one possibility that the two nuclei may stick together and from an α-particle. But such an α-particle would have a very high energy content. This energy might be used for the emission of radiation. And if it is so used, the α-particle may stick together permanently. Emission of radiation takes, however, too long a time and it is more probable that before a γ ray could be emitted, the α-particle would fly apart again. It might fly apart into two deuterons, i.e., one would get back the original nuclei and the net effect would be merely a collision in which the deuterium nuclei have deflected each other. Other reactions are also possible. Each of the deuteriums consisted of a neutron and a proton. Thus, in the moment of their collision, we have two neutrons and two protons and either one of these four particles might fly off. Thus we might get as a result of the reaction a proton and a H^3 or a neutron and He^3. Both these reactions have been observed. They proceed with a considerable evolution of energy but the initial particles must also have had appreciable kinetic energy, otherwise the reaction becomes quite improbable, since the particles cannot approach sufficiently close.

Another possible reaction would be the formation of a nucleus consisting of two protons and of another nucleus consisting of two neutrons. These nuclei are not stable. Finally, three or four particles may be obtained as a final product, such as a deuteron, a neutron and a proton, or two neutrons and two protons. These reactions require high amounts of kinetic energy in the initial particles and have not been observed. The reaction

$$H^2 + H^2 \longrightarrow He^3 + n$$

is a good source of neutrons. The resulting neutrons have approximately 2.5 MV. Another neutron source is the

$$Li^7 + H^1 \longrightarrow Be^7 + n$$

reaction. If Li is bombarded by hydrogen, a great number of reactions are observed due to the fact that Li has two isotopes, namely Li^7 and Li^6, and these isotopes may react in various ways with the hydrogen nuclei. These reactions could be disentangled eventually by using separated Li samples, in which only Li^7 or Li^6 was present. The particular reaction quoted above,

$$Li^7 + H^1 \longrightarrow Be^7 + n$$

is endothermic, approximately 2MV kinetic energy being needed in the protons for the reaction to proceed. The excess energy of the incident protons over the threshold will turn in greatest part into kinetic energy of the neutron. By controlling the proton energy one might get neutrons of given energies. This reaction is actually the best way to produce fairly monoenergetic neutrons in the range of a few tenths of a MV.

The reaction

$$Be^9 + H^2 \longrightarrow B^{10} + n$$

is a very copious source of neutrons if deuterons of 3 MV or more are used. Such deuterons may easily be produced in cyclotrons. The reaction is strongly exothermic and gives therefore neutrons of very high energies (around 10 MV). The neutrons so produced do not ever have a well defined energy. The B^{10} produced in the reaction may be left behind in various states of excitation, and the neutrons carry away the varying amounts of energy that have not been used up in exciting B^{10}. Eventually, the excitation energy of B^{10} is emitted in the form of γ radiation. It is to be noted that in the d+d reaction no excitation occurs because the very light nuclei obtained in this reaction (for instance H^3) do not have excited levels in the neighborhood of 3 or 4 MV, which is the energy available in this reaction. This energy must therefore be carried off as kinetic energy of the reaction products and one obtains neutrons of well-defined energies.

LA 24 (5)

September 28, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

First Series: Terminology Lecturer: E.M.McMillan

LECTURE V: RADIOACTIVITY

Historically the first nuclear transformation observed was radioactivity. It was noticed that some minerals continually give off certain types of radiations which were named α, β and γ - radiations. Their nature was later discovered. The α-radiation was identified with He^4 nuclei, the β-radiation with electrons, and the γ-radiation with quanta of electromagnetic radiation. In a radioactive process a nucleus undergoes a spontaneous change, in which it may emit a part of its own substance. The γ-radiation however corresponds merely to a settling down of a nucleus from a higher energy state into a lower, more stable state. It usually follows some other reaction and is accompanied by no chemical change of the substance.

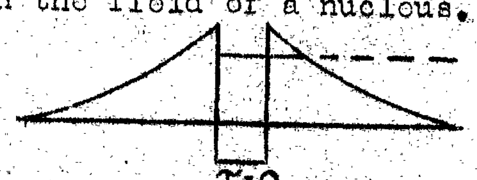
When an α-ray is emitted, the ejected particle carries two charges and the nucleus is left with two charges less, that is, it moves down by two places in the periodic system. An example is the reaction

$$Ra^{226} \longrightarrow Rn^{222} + He^4$$

in which the element Radium, which is analagous in its properties to Barium, is transformed into the noble gas Radon. This reaction is balanced in the same manner as the artificial reactions mentioned earlier but an essential difference is that it occurs spontaneously. Following the above reaction, Radon emits in turn an

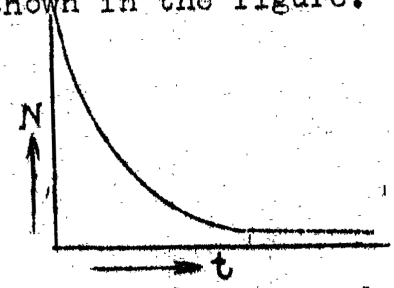
α-particle and transforms into an element which chemically behaves like Polonium. It is actually an isotope of Polonium. Another isotope of Polonium (which is usually meant when one talks of Polonium) is obtained after a series of further transformations in which partly α, partly β, particles are emitted. The fact that two substances behave chemically in an identical way and yet differ in their radioactivity and in other nuclear properties led to the discovery of isotopes. We have seen that the radioactive transformation of Radium is followed by a chain of further transformations. In some of them α-particles are emitted and the remaining nucleus is moved down two places in the periodic system. In others, the β-active substances in the natural radioactive series, a negative electron is emitted and in order to conserve the charge, the element has to move up by one place in the periodic system. Radium itself is derived from other radioactive substances. In the particular series to which Radium belongs, all elements are descended from Uranium which has a very long life of several billion years. The other elements in this series have much shorter lives but they are replenished from the original stock of Uranium.

It is remarkable that the α-active substances take such a long time to emit the α-particle. Radium for instance, has a life of about 2000 years. The figure shows the potential energy of an α-particle in the field of a nucleus.



At great values of r, the interaction between the α-particle and the rest of the nucleus is electrostatic repulsion. At small distances, the stickiness of the nucleus will cause the potential energy to be lowered, and near r = 0 we find a potential well. The horizontal line not far from the top of the well represents the energy level which the α-particle occupies within the nucleus. This energy level in a radioactive nucleus is higher than the potential energy at very high r values. This must be so because we know that at high r values the α-particle may appear carrying considerable kinetic energy, which kinetic energy must be derived from the energy that the α-particle originally possessed in the nucleus, but in order that the α-particle should escape the nucleus (moving along the dotted line in the figure) it must cross a region of high potential energy. According to classical mechanics this is impossible. In quantum mechanics the fuzziness of particles makes penetration through the potential barrier possible but the probability of the particle leaking through is very small. This is true in particular if the potential barrier is high. One finds that the average time which the α-particle is supposed to spend in the nucleus before leaking out by the quantum mechanical method is in fair agreement with the times actually observed. The sensitive dependence of this time on the height of the potential barrier explains why radioactive decay times vary from a small fraction of a second to several billion years.

The mathematical law of radioactive decay is based on the fact that each radioactive nucleus has a constant probability of decay. It is to be noted that this probability of decay is independent of the time the particular nucleus has already lived. It follows that in a bunch of radioactive nuclei, the number of nuclei decaying per second is proportional to the number of nuclei present. If one plots the number of nuclei as a function of time, one obtains a curve shown in the figure.



The curve is an exponential function and the number of nuclei is given by the formula

$$N = N_o e^{-\lambda t}$$

N_o is the original number of neutrons (which were present at $t=0$) and λ is the reciprocal of the so-called mean life. It is understood that some nuclei live shorter, and a few, considerably longer, than this mean life. It is more usual to describe the radioactive decay by the half-life which is given by

$$\text{half-life} = 0.693/\lambda$$

The half-life is the time in which half of the original number of the nuclei has decayed.

In discussing the β-decay we shall consider an artificially radioactive substance. Artificially radioactive nuclei obey the same laws as naturally radioactive ones; but instead of being encountered in nature, they have to be produced by bombarding a naturally occurring nucleus by a light nucleus. In this way, many β-active substances have been produced while artificial α-activities are very rare. A well-known artificial β-active substance is obtained by bombarding the one stable isotope of Sodium by Deuterium

$$Na^{23} + H^2 \longrightarrow Na^{24} + H^1$$

One obtains the Na^{24} nucleus which decays according to the equation

$$Na^{24} \longrightarrow Mg^{24} + e^-$$

In this equation we have written e^- instead of β in order to show that a negative electron is emitted. The nucleus produced in the reaction, Magnesium, has one more charge than Sodium and thus the total charge is conserved.

It has been stated that in a nucleus there is no room for electrons. How is it then possible that the nucleus should emit an electron? It must be postulated that the electron is created in the β-disintegration process. Other processes are indeed known in which particles are created. One example is the creation of electron-positron pairs from radiation. The elementary creation process in β-activity is the

$$n \longrightarrow p^+ + e^-$$

reaction. The energy delivered to the electron is partly due to the mass difference between the proton and the neutron and partly it is drawn from the energy store of the β-active nucleus in which the neutron was originally formed. The above simple process is exothermic. It is assumed that the neutron itself is β-active. This activity however takes a longer time than other processes by which neutrons can disappear and therefore the β-activity of free neutrons has as yet not been observed.

Another kind of β-active nucleus is formed in the reaction

$$N^{14} + H^2 \longrightarrow O^{15} + n$$

The O^{15} nucleus decays according to the equation

$$O^{15} \longrightarrow N^{15} + e^+$$

This time a positive electron is emitted. This kind of activity is a little rarer than the one discussed above. It does not occur in the naturally radioactive series. The elementary reaction in this case is

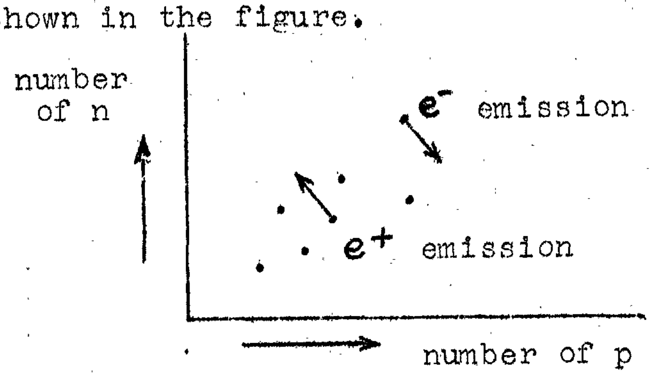
$$p^+ \longrightarrow n + e^+$$

This time the elementary reaction is endothermic. This means, that the proton is stable and a positron can be emitted only if the energy store of the β-active nucleus can supply an energy sufficient to overcome the mass difference between proton and neutron.

Sometimes it happens that a nucleus instead of emitting a positron absorbs one of its own electrons. As a rule this electron will be one of an inner layer in an atom, that, a so-called K-electron. This process requires a somewhat smaller energy than the emission of a positron because of the mass (and corresponding energy) of the positron and electron. In principle it would be possible for a nucleus to capture a positron. But even the capture of an electron is an improbable process in spite of the fact that the electron is near to the nucleus all the time. It practically never happens that a positron on its brief passage near a nucleus is absorbed.

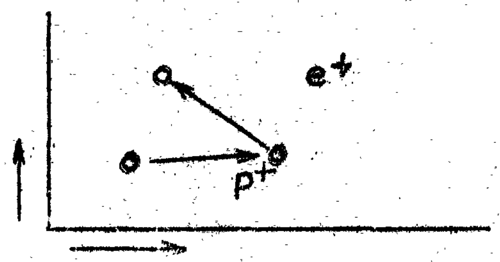
The mathematical description of the β-decay is the same as of the α-decay. The half-lives range from a fraction of a second to hundreds of years. Slowness of the β-process however cannot be explained by the presence of a potential barrier since the electron has a long enough wave length to leak through nuclear potentials in exceedingly short times. The long lives of β-processes (even a fraction of a second is a long time on a nuclear time scale) must be explained by an inherent unwillingness for the neutrons and protons to transform into each other. We must accept the fact that the elementary creation processes mentioned above have a low probability and the corresponding reactions proceed at a slow rate. This also explains the fact why bombardment of nuclei by electrons does not produce typical nuclear reactions. The only action of electrons on nuclei which has been observed is due to the electric field of the electron which gives rise to similar excitation processes as absorption of γ-rays.

β transformations are frequently represented in nuclear charts as shown in the figure.



The ordinate gives the number of neutrons, the abscissa the number of protons. Dots in the diagram represent nuclei, diagonal arrows represent β-transformations. The β-transformations actually limit the region in this diagram in which stable nuclei can be found. If a nucleus contains too many protons, it will emit one or more positrons and the nucleus will thereby return to the belt of stable nuclei. If too many neutrons are present, electron emissions will accomplish the same thing. These emissions must happen in single steps. Simultaneous emission of two electrons or two positrons is exceedingly improbable, as will be seen from the fact that the probability of such a process must be the square of the probability of a single emission process which we have stated is of itself improbable. No such double process has ever been observed. It can happen however that a nucleus may emit either an electron or a positron. This occurs when the nucleus can decrease its energy either by transforming a neutron or by transforming a proton into a neutron.

Charts of the kind shown above are useful in following any kind of nuclear reaction. In the part of a chart shown in the figure below, two reactions are shown. In the first a proton is added; a β-active nucleus is obtained and a positron is emitted subsequently.



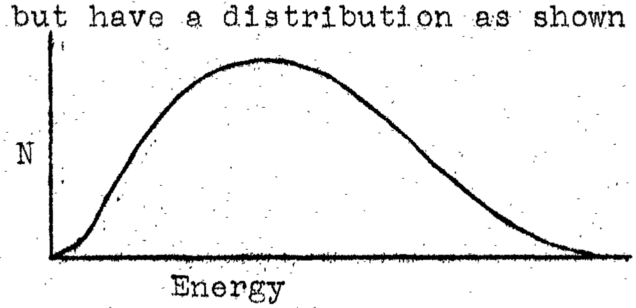
September 30, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

First Series: Terminology Lecturer: E. M. McMillan

LECTURE VI: I) β-DECAY II) NEUTRON REACTIONS

I) In a nuclear process in which a β-particle (electron) is emitted, the particular nucleus goes from one definite energy state to another definite energy state. Hence one would expect the emitted β-particle to possess a definite kinetic energy. Experimentally, however, it is found that the β-particles are not homogeneous in energy but have a distribution as shown in the following figure.



The upper limit of the energy corresponds to the value expected by the law of conservation of energy. Several explanations for the lack distribution in energy have been suggested, but as yet no completely satisfactory theory has been given. One may suppose that a γ -ray is emitted during the process to account for the discrepancy in energy, but cases are known definitely where no γ-ray is emitted. It has been suggested that energy conservation does not hold. However the preferred present hypothesis is that energy conservation is valid and that part of the energy is emitted as undetected radiation. This radiation is supposed to consist of particles possessing some remarkable properties: They are uncharged and possess very small mass, hence they cannot be observed. In order to conserve angular momentum, one must ascribe to them an angular momentum of 1/2 quantum unit. This extraordinary particle has been called the neutrino.

II) The neutral character of the neutron permits it to approach a nucleus without having to overcome any repulsive forces. As a result, neutrons with slow velocities are quite as effective in initiating a nuclear reaction as fast moving ones. Thus if one were to plot a yield curve as a function of the neutron energy, one would expect as a rough approximation that the curve would be a constant. Actually the cross-section (i.e., the probability) of the reaction increases with <u>decreasing</u> velocity of the neutron. One may see that this fact is plausible by recalling that the wave length associated with the neutron is <u>inversely</u> proportional to its velocity. Thus the slow neutron is not localized as much as a fast one and consequently has a greater probability to strike a nucleus.

A plot of the cross-section σ as a function of neutron velocity V is shown in the figure below. For large values of v the cross-section approaches the geometrical cross-section of the nucleus.

The above curve however is not the complete story. For it is found in some cases that if a beam of neutrons have a certain definite velocity (within a rather small range) the cross-section is much larger than would be expected on the above picture. This is the so-called phenomenon of <u>resonance.</u> The places on the velocity (or energy) scale where these resonances occur depend on the particular nucleus. The fact that gadolinium and cadmium have such exceedingly high cross-sections for slow neutrons may be explained by the presence of resonances at velocities corresponding to thermal energies. Experimental methods are known for determining the position of these resonances.

The large cross-sections for neutrons of thermal velocities make it desirable to have an adequate source of slow neutrons. Most of the neutrons from nuclear reactions have energies exceedingly large compared to thermal. It is desirable to "slow down" or moderate them. A satisfactory method that is used is to have them make collisions with light nuclei, inasmuch as the energy transfer proceeds best when the two colliding particles have the same masses. Hydrogeneous materials such as paraffin are often used. Graphite is also used; although it is not as good a moderator as paraffin, the loss of neutrons by capture is less.

There are several distinct types of nuclear reactions induced by neutrons: (1) simple capture, usually followed by the emission of a γ-ray, the resulting nucleus being an isotope of one mass unit larger than the original nucleus; (2) inelastic collision; the internal energy of the nucleus is changed. This reaction may be regarded as an absorption of a neutron by a nucleus and a subsequent re-emission of a neutron. (3) The (n,p) reaction: a neutron is absorbed and a proton is emitted. This reaction proceeds with fast neutrons, although a slow neutron reaction is known for nitrogen. Here the product nucleus has an atomic number one smaller than the bombarded nucleus. (4) The (n,α) reaction: examples of this are

$$Li^6 + n^1 \longrightarrow He^4 + H^3$$
$$B^{10} + n^1 \longrightarrow Li^7 + He^4$$

These reactions proceed with slow neutrons. For heavier nuclei, the α-particle must overcome a potential barrier; hence fast neutrons are required. (5) The (n, 2n) reaction in which two neutrons are emitted requires fast neutrons. (6) For the sake of completeness, elastic collisions are included. Here the internal energy of the irradiated nucleus is unchanged.

LA 24 (7)

October 5, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

First Series: Terminology — Lecturer: E.M. McMillan

LECTURE VII: NUCLEAR FISSION

The word fission means splitting or breaking apart. We are concerned here with a process in which a nucleus splits into two more or less equal parts. The product nuclei are fairly heavy. Only such fission processes are known in which the initial nucleus is one of the heaviest nuclei.

The reason for the fission process is the electrostatic repulsion between protons. In highly charged nuclei the potential energy of this repulsion is so great that by breaking apart the nucleous can liberate energy. To illustrate this, we plot the mass defect as a function of the atomic number for stable nuclei.

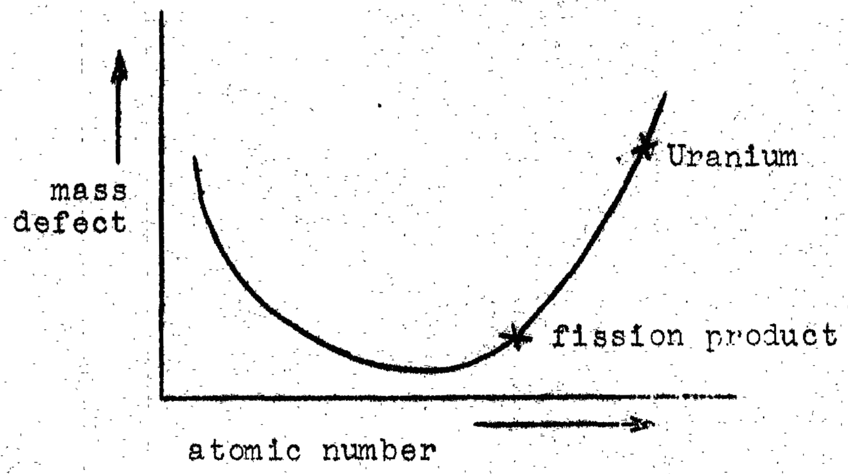


Fig. 1

The mass defect is a measure for the potential energy per particle in the nucleus. One obtains it by subtracting from the nuclear mass the nearest integer (which is the number of neutrons and protons in the nucleus and has been called the mass number) and then dividing by mass number. By going from Uranium to the fission products, this potential energy per unit particle has considerably decreased and in the fission process an energy of the order of 100 MV may be liberated.

The possibility of such energy liberation has been known for a long time but it has been considered unlikely. Actually fission occurs when Uranium is bombarded by neutrons. The fission products are radioactive. These activities have been observed, but for a considerable time they were not interpreted as due to fission products because of the preconceived idea that the occurrence of the fission process is practically impossible. Finally Hahn observed that one of the activities is definitely due to Barium. Since this nucleus has only little more than half the mass of Uranium, it became evident that the Uranium nucleus did break into large fragments. Physicists then started to look for highly ionizing particles. It could be predicted that fission will give rise to sudh particles, because the great amounts of energy liberated in the process must eventually be dissipated in the form of ionization processes. The highly ionizing particles were easily found by letting fission proceed in an ionization chamber in which the ions formed are drawn out by an electric field and collected on the electrodes. The pulses and corresponding ionizations obtained were many times stronger than those produced by α-particles which up to that time were the most strongly ionizing particles.

A picture explaining the fission process is based on the droplet model of heavy nuclei. Like droplets, heavy nuclei are supposed to be held together by surface tension. Surface tension acts like an elastic skin. It can be explained by the tendency of the constituent particles of the droplet to get in as close touch with each other as possible. On the surface the particles do not make as many contacts as they would in the interior. Thus a surface means a positive potential energy, and there is consequently a tendency to form as small surfaces as possible.

The charge of the nuclei acts in the opposite manner from surface tension in trying to pull the nucleus apart. As long as the droplet is spherical (as shown in Fig. 2) the charge, being symmetrical, will not be able to produce motion. If, however, the nucleus assumes an ellipsoidical shape, the charges will accumulate near the ends of the ellipsoid (as shown in Fig. 3).

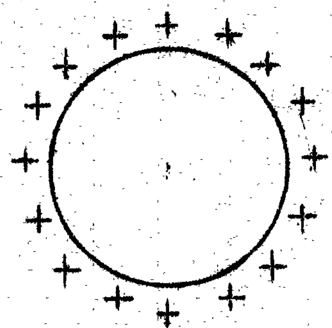


Fig. 2

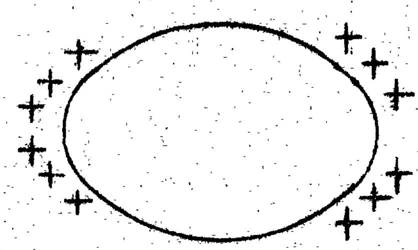


Fig. 3

The charge will then tend to cause further elongation and might split the droplet. This effect was observed in the case of water drops. The larger the charge, the smaller elongation will cause the droplet to break up. This picture explains therefore why fission can be expected with greatest probability in the most highly charged nuclei.

In Uranium and Thorium, fission has been induced by neutron capture. If the neutron is captured, its binding energy becomes available. This energy is as a rule eventually emitted in the form of γ-radiation. For that radiation, however, considerable time is needed. The first effect of the binding energy is to set the particles within the nucleus into motion. An oscillation may set in and may give to the nucleus, temporarily, an elliptical shape which is well known to occur in the oscillation of droplets. Then fission may result.

Uranium has two principal isotopes, U^{238} and U^{235} (a third isotope U^{234} is very rare). The U^{238} nucleus is the parent of the Radium series. U^{235} is that of the Actinium series. Both these nuclei undergo fission. In U^{238} the fission has a threshold of a little more than 1 MeV. The fission cross-section, σ_f, varies with the energy of bombarding neutrons as shown in the following figure.

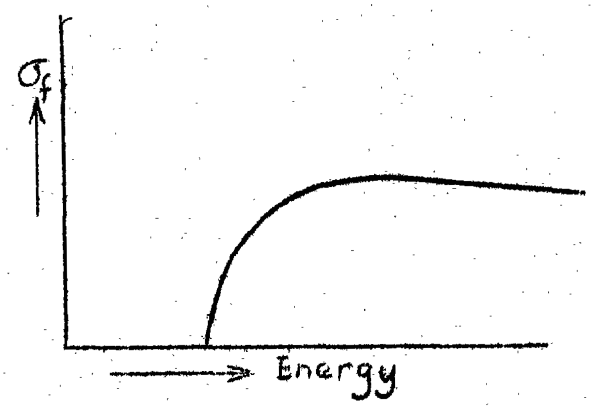


Fig. 4

The threshold energy is the amount of energy needed, beyond the binding energy of neutron, to carry the nucleus beyond the stability point, that is, to give to the nucleus a sufficiently ellipsoidical shape. Since this energy is only needed to overcome a potential barrier, the question arises why by the leaking through process fission does not occur spontaneously. The answer is twofold: first, the barrier is high and broad; and second, spontaneous fission does occur, but at an exceedingly slow rate.

There is no threshold in the fission U^{235}. This nucleus obtains by neutron capture an even number of neutrons and since nuclei with an even number of neutrons have a stronger binding energy, more energy is liberated in this neutron capture process than in the capture of U^{238}. Therefore, enough energy is available to start the fission process even if the neutron brings no kinetic energy along.

Fission in Thorium behaves in a way which is similar to fission in U^{238}.

The β-activity of the fission products may be explained by considering the plot of the stable nuclei.

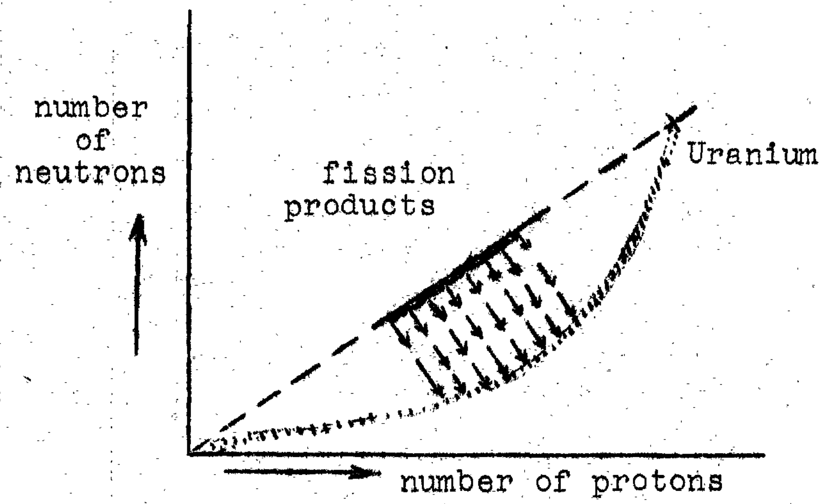


Fig. 5

In this figure, the dots as usual represent nuclei, their ordinates and abscissae giving the number of neutrons and number of protons contained in the nucleus. If Uranium splits into two more or less equal fragments, the corresponding points indicated by the solid line are considerably above the region of stable nuclei. The fission products will then get back to that region by a series of β-decays, indicated in the figure by small arrows. Many chains of this kind have been found. Most of these activities were not known previously. However, some of them near the end of the chains and lying close to the region of stability are known artificial activities. These activities have helped to identify the mass numbers of these radioactive chains. The atomic numbers could, of course, be found by investigating the chemical properties.

While the product nuclei in fission have approximately equal weight, there is a marked tendency for a slight dissymmetry. In the most probably fission process, the ratio of masses of the fission products is distinctly different from unity. In spite of many attempts, no explanation has been found which is quite simple and convincing.

October 7, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Second Series: Radioactivity Lecturer: E. Segre

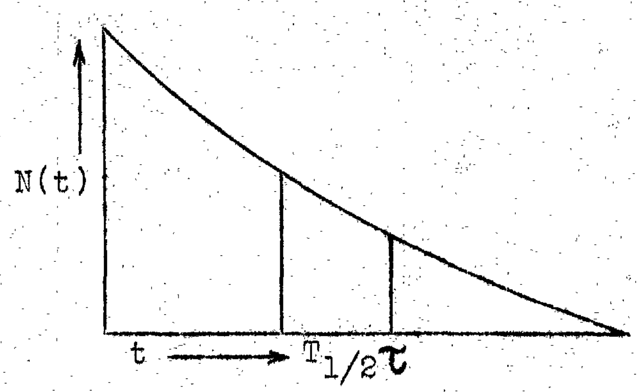
LECTURE VIII: THE RADIOACTIVE DECAY LAW

Each atom has a probability λdt to decay in time dt. The coefficient is called the decay constant, and of course varies with various substances. Alpha, beta and gamma emissions all follow the same decay law. From this it follows that change of number of atoms dN in time dt is $-N\lambda dt$, where N is the number of atoms present at time dt. The fact that N is an integer causes difficulties with the definition of the differential dN. Mathematically these difficulties can be obviated by using appropriate averages. However, the difficulties actually do arise in the experiments when the number of decays have to be established by counting. This point will be discussed later.

The differential decay law given above may be integrated and gives

$$N(t) = N(0)\, e^{-\lambda t}$$

where N(t) is the number of atoms present at time t and N(0) is the original number of atoms which were present at $t = 0$. In the following figure N(t) is plotted against the time. In addition, this figure shows the time τ which is called the mean life and which may be defined as the time at which the number of atoms have decreased to 1/e of their original value. Also shown in the figure is the half life, or period, designated by $T_{1/2}$ or simply T. This is the time in



which the number of atoms has decreased to 1/2 of their original value. It is seen from the radioactive decay formula that $\tau = 1/\lambda$. Also one has

$$e^{-\lambda T} = 1/2$$

from which it follows that

$$\lambda T = \ln 2 = .6931 .$$

The designation mean life or τ is justified by the fact that it is actually the average of the time an atom lives before decaying. In fact the number of atoms which have lived for a time between t and $t + dt$ is $N(t)\lambda dt$. Multiplying by t, integrating from 0 to a, and dividing by the original number of atoms, one gets for the mean life

$$1/N(0) \int_0^a tN(t)\lambda dt = 1/\lambda$$

It is easy to verify that this expression is $1/\lambda$ which by definition is equal to τ_1.

The number of atoms decaying per unit time is defined as <u>activity</u>. It is equal to $N(t)\lambda$. Activities are measured in units of curies. A curie corresponds to 3.7×10^{10} disintegrations per second. The reason for the choice of this number was that it was supposed to be the number of disintegrations per second occurring in one gram of radium. Recent measurements seem to show that the number of disintegrations in radium is actually somewhat lower (the latest value is 3.46×10^{10} dis/sec). But it is better to retain the number 3.7×10^{10} dis/sec as a fixed unit. In medicine a different meaning is often adopted for the expression "radium equivalent." Thus a milligram radium equivalent may mean an amount of substance which gives rise behind a 10 mm lead shield to the same density of ionization as would be caused by 1 mg of radium. The reason for such a definition is that the biological effects of radioactive substances are due to their ionizing power.

The change of the amount of radioactive substance with time may be quite complicated if the radioactive substance is itself a product of radioactive decay or even more particularly if the radioactive substance is a member of a long radioactive chain. How complicated such chains may become can be illustrated by the example of the Uranium family. The following table is essentially taken from the Handbook of Chemistry and Physics, 27th Edition, page 315.

The data given in the handbook are the results of an international agreement on the best available information in 1930. In the meantime, changes are necessary. These changes have been incorporated in the following table. These are not the only changes needed.

INTERNATIONAL TABLE OF THE RADIOACTIVE ELEMENTS AND THEIR CONSTANTS

Name	Symbol	Half period T	Radiation	Isotope
Uranium and Radium Series				
Uranium I	U_1	4.56×10^9 yrs.	α	U
Uranium X_1	$U\text{-}X_1$	24.6 days	β	Th
Uranium X_2	$U\text{-}X_2$	1.15 min.	β	Pa
Uranium II (Brevium)	U_{II}	2.68×10^5 yrs.	α	U
Ionium	Io	6.9×10^4 yrs.	α	Th
Radium	Ra	1690 yrs.	α	Ra
Radon (Radium emanation, Niton)	Rn	3.85 days	α	Rn
Radium A	Ra-A	3.0 min.	α	Po
Radium B	Ra-B	26.8 min.	β	Pb
Radium C	Ra-C	19.5 min	99.97% β	Bi
Radium C'	Ra-C'	1.44×10^{-4} sec	α	Po
Radium D (Radiolead)	Ra-D	16.5 years	β	Pb
Radium E	Ra-E	5.0 days	β	Bi
Radium F (Polonium	Ra-F	136 days	α	Po
Radium Ω' (Lead)	RaΩ' Pb^{206}	....	....	Pb
Radium C	Ra-C	....	.03% α	Bi
Radium C" (Radium C_2)	Ra-C"	1.4 min.	β	Tl
Radium Ω"	Ra Ω"	....	....	Pb

The change in the number of atoms N_1 of the first member of a radioactive series obeys the simple differential decay law mentioned in the beginning of the lecture

$$dN_1 = - \lambda_1 N_1 dt$$

where λ_1 is the decay constant of the first member of the series. The change in N_2, the number of atoms of the second member of the series is

$$dN_2 = \lambda_1 N_1 dt - \lambda_2 N_2 dt .$$

The first term of the right hand side is the increase in the number of atoms N_2 due to the decay of the first member of the radioactive series. Similarly, we can write for the third member

$$dN_3 = \lambda_2 N_2 dt - \lambda_3 N_3 dt$$

Similar equations hold for further members of the series. The system of equations thus obtained may be solved by writing

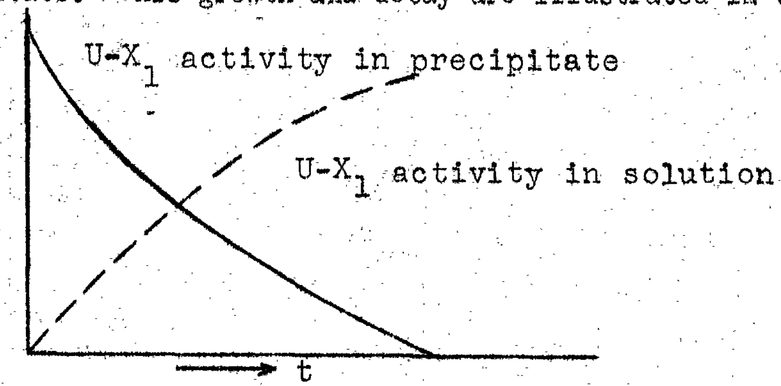
$$N_1 = a_{11} e^{-\lambda_1 t}$$

$$N_2 = a_{21} e^{-\lambda_1 t} \quad a_{22} \quad e^{-\lambda_2 t}$$

$$N_3 = a_{31} e^{-\lambda_1 t} \quad a_{32} \quad e^{-\lambda_2 t} \quad a_{33} \quad e^{-\lambda_3 t}$$

The coefficients a_{11}, a_{21}, a_{31}, etc., may be determined by two operations: first, substituting the expressions for N_1, N_2, N_3, into the differential equations, governing the time rate of change of these quantities, and second, writing the expressions for N_1, N_2, N_3, etc., for $t = 0$, and equating these quantities to the known initial amounts of the radioactive substances. In this way a necessary and sufficient number of equations is obtained to find all the coefficients. Solutions for the time dependence of N_1, N_2, etc., in some of the cases of practical importance in the Ra family are tabulated in the book of Mme. Curie.

A simple consequence of the equations discussed is the radioactive equilibrium. If, e.g., a uranium solution is left standing for a time long compared to the half-life of the first daughter product U-X_1 (i.e., long compared to 25 days), then a stationary state will establish itself in which the same number of U-X_1 nuclei decay and are formed per unit time. This is the case if the activities of U_1 and U-X_1 become equal, or if the ratio of the number of uranium atoms to the number of U-X_1 atoms is the same as the ratio of the period of the uranium atom to that of the U-X_1 atom. This simple concept of radioactive equilibrium is based on the fact that the original mother substance U_1 has an exceedingly long life and its depletion can be neglected during the growth of the daughter substance. If U-X_1 is precipitated out of a uranium solution, all the U-X_1 will be formed in the precipitate and none in the solution. The solution, however, retains all of the mother substance U_1. Subsequently, $U\text{-}X_1$ will grow in the solution, while it will decay in the precipitate. This growth and decay are illustrated in the following figure.



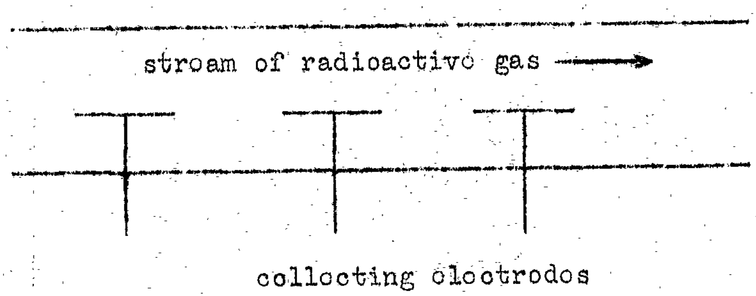
The decay of $U\text{-}X_1$ in the precipitate follows the simple exponential radioactive decay law. The growth of $U\text{-}X_1$ in the solution can be obtained from the fact that chemical separations must have left the total activity in precipitate and solution unchanged. Therefore, the two curves in the figure must add up to a constant. These observations were originally due to Rutherford around 1900. Before he cleared up the situation, it was very confusing to chemists that a solution purified from a radioactive material one day would show the same activity again the next day. The figure shown above has been chosen by Lord Rutherford for his escutcheon.

Radioactive equilibrium involving substances of longer periods can be studied in rocks. Thus the ratio of half-lives of U_I and U_{II} has been determined from the ratio of abundances of these atoms in uranium-carrying minerals. Since U_I and U_{II} have the same properties, their ratio has to be determined by the mass spectrograph.

An absolute determination of the life time of uranium itself cannot be carried out by waiting for its decay. In this case, as in other cases of long periods, the usual procedure is to make a thin film of known weight of uranium and count the α-particles emitted. In this way, one can find the number of disintegrations per second, provided that the film is thin enough so that all the α-particles emitted by uranium can actually get through the film. The same method is used for determining the half-life of radium. In this case, however, the oldest radium samples (about 40 years old) begin to show an activity slightly smaller than they had originally.

Periods in the range between a few seconds and a year may be conveniently determined by following the change of activity with time. Most of the known artificial activities fall into this range. This is due to the fact that activities with very short or very long lives are hard to detect. Even where the simplest determination of life time is feasible, a precise determination of the life time is difficult. Thus the best known period of a naturally active element, that of radon, is known to a precision of .05%, while the period of the well known artificially active nucleus P^{32} is known to .2%. The usual accuracy of half-lives is less than 1 or 2%. This experimental uncertainty may give rise to considerable errors if amounts of a radioactive substance are to be calculated from the decay formula from measurements taken long after the time in which one is interested.

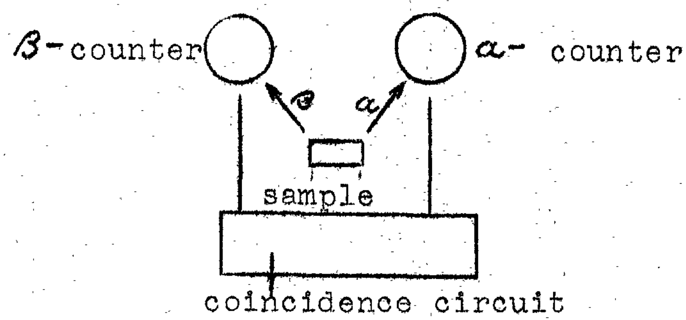
Special methods are needed to measure periods which are shorter than a second. One method applicable for radioactive gases is to let the gas with a known velocity stream in a pipe, past a series of collecting electrodes, as shown in the figure.



The current collected by these electrodes is proportional to the ionization caused in their neighborhood by the radioactive gas and this in turn is proportional to the number of radioactive atoms remaining in the stream. From this measurement and the known velocity of the gas, the decay period may be calculated.

A similar method has been used for known gaseous substances. But instead of the velocity of the gas stream, the recoil velocity of the radioactive atoms was utilized, the recoil being due to the α-emission by which the short lived substance was formed. This method is not reliable because instead of single atoms, groups of atoms or crystallities may recoil, thus making the recoil velocity uncertain.

A more successful method utilizes coincidence counters.



Let us suppose that a parent substance emits β-rays and thereby transforms into an α-active substance of very short period. The substance is placed next to an and to a β-counter and the counters are connected to a coincidence circuit which will give counts only if the two counters are activated within a time t_c apart. Then the number of coincidence counts to be expected is

$$C = -C_{max} \int_{t_c}^{0} e^{-\lambda t} \lambda dt = C_{max} (1 - e^{-\lambda t_c})$$

Here C_{max} is the number of counts to be expected if t_c is chosen as a very long time. This quantity is multiplied by the integral of the activity $\lambda e^{-\lambda t}$ of unit substance over the period t_c. Thus C gives the number of counts obtained if only those α-disintegrations are effective which occur within a time t_c after the β-disintegrations. Repeating this experiment for various times t_c one can find λ from the variation of C. This experiment has been carried out for the Ra-C — Ra-C' pair of radioactive substances. It is claimed that this method is capable of measuring half-lives between 10^{-1} and 10^{-7} seconds.

LA 24 (9)

October 12, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Second Series: Radioactivity   Lecturer: E. Segre

LECTURE IX: 1) THE RADIOACTIVE DECAY LAW (continued) 2) FLUCTUATIONS 3) PASSAGE OF PARTICLES THROUGH MATTER

I. THE RADIOACTIVE DECAY LAW

One application of the radioactive decay law is the determination of the age of the earth. This can be done by determining lead in uranium ores. If originally no lead was present in the ore and if theore has remained undisturbed since its formation, the lead formed by radioactive decay will give a measure of the time that has passed since the formation of the ore. Another similar method measures the helium formed by radioactive decay and remaining in the ore. From the age of the oldest ores, the presumable age of the earth can be determined. Values of the order of 10^9 years have been obtained.

II. FLUCTUATIONS

Radioactive measurements are usually made by counting disintegrations. The fluctuations occurring in these counts introduce errors into the measurements. We shall investigate the influence of these fluctuations.

The simplest case is if the substance under investigation does not decay appreciably during the time of the measurement. We shall consider such a case. Assume that the activity of the substance corresponds to m counts per second. The probability of finding n counts in a second instead of the expected value m is denoted by p_n [1]. To find p_n we subdivide the second into k parts and make the subdivision fine enough so that the probability of finding

1) The original calculation of these probabilities was due to Bernouilli and Piosson.

two counts in one subdivision is negligible. To find one count in such a subdivision has a probability m/k. To find n particles in the first n divisions and to find none in the following $k-n$ subdivision has the probability

$$(m/k)^n \quad (1 - m/k)^{k-n}$$

To find n counts distributed in any manner among the k intervals, we must multiply this last expression by the binomial coefficient $\binom{k}{n}$ i.e., the number of ways in which n intervals can be chosen among k. Thus we obtain for p_n

$$p_n = \binom{k}{n} (m/k)^n (1 - m/k)^{k-n}$$

If we now let k go to infinity, the first factor in the above expression becomes

$$\binom{k}{n} = \frac{k(k-1)\ldots(k-n \;\; 1)}{n!} \underset{k\to}{\longrightarrow} \frac{k^n}{n!}$$

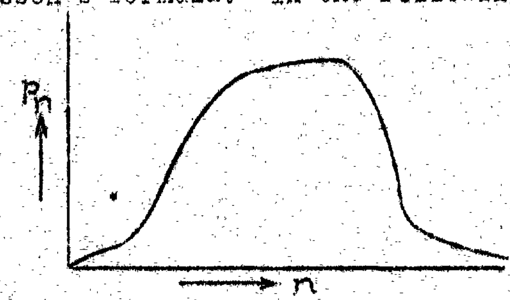
whereas the last will be

$$e^{-m}$$

Therefore the formula simplifies to

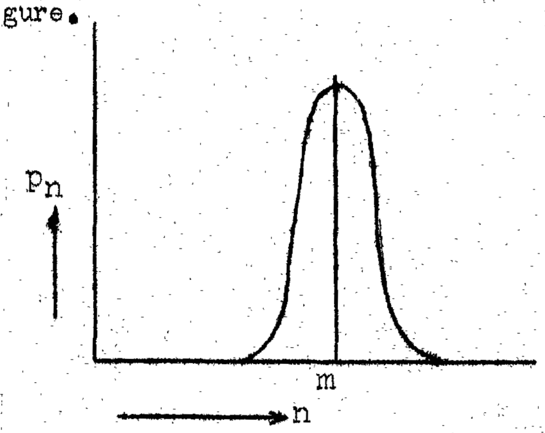
$$p_n = \frac{m^n}{n!} e^{-m}$$

which is known as Poisson's formula. In the following figure a plot of p_n



against n is shown. Of course, Poisson's formula is defined only for integral values of n. But after plotting these integral values one can interpolate the rest of the curve. It is usual in determinations of half-lives to plot the frequency of occurrence p_n of a given number of counts per second n against that number n and to compare this plot with Poisson's formula. If n can be so adjusted as to give a satisfactory fit, one has thereby obtained the desired value of the counts per second and has at the same time checked the proper statistical functioning of the counter.

If m is very large, the plot of the Poisson formula looks as indicated in the following figure.



The curve has now a rather symmetrical sharp maximum around the value $n = m$ and goes quickly to zero when the deviation of n from m becomes appreciable. In this case, Poisson's formula may be replaced by the Gauss approximation

$$p_n = \frac{1}{\sqrt{2\pi m}} e^{-(n-m)^2/2m}$$

This formula can be derived by assuming that $n - m$ is small compared to m, using the Stirling approximation for the factorial, taking the logarithm of p_n and expanding into a Taylor series near the maximum $n = m$.

The width of the curve in the above figure can be defined by choosing a value w in such a way that one half the area under the curve lies in the region where $|n - m|$ is smaller than w. This means that the deviation of m from n has the same probability of being greater or of being less than w. The value of w can be shown to be

$$w = 0.6745 \sqrt{m}$$

Another interesting quantity is the average value of $(n-m)^2$. This is

$$\sum_n p_n \quad (n - m)^2 \sim$$

denoting $n - m$ by x and the average value of its square by $\overline{x^2}$ and replacing by an integral (which is permissible for the large m values considered here) we obtain

$$\overline{x^2} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi m}} x^2 e^{-x^2/2m} dx = m$$

Thus the root mean square of $|n - m|$ is $\sqrt{m}$. If we have counted m events in a time t, we have an even chance that by repeating the measurement many times the average result will be between $m - 0.6745\sqrt{m}$ and $m + 0.6745\sqrt{m}$ or outside that interval. $0.6745\sqrt{m}$ is called the probable error and $\sqrt{m}$ the standard deviation. The standard deviation increases with m, the relative standard deviation $\sqrt{m}/m = 1/\sqrt{m}$ decreases with m. If we count 100 events the relative standard deviation of our measurements is 10%, if we count 1000 events the relative standard deviation is 3.1%, etc.

It is sometimes important to know the probability of an error equal to a multiple times the standard deviation. These probabilities are tabulated, e.g. in the Handbook of Physics, 27th edition, page 199.

The question sometimes arises what conclusions can be drawn about the decay constant if one finds no counts during the time of observation. It follows from Poisson's formula that the probability of finding no counts in an interval in which m counts should have been expected is

$$p_o = e^{-m}$$

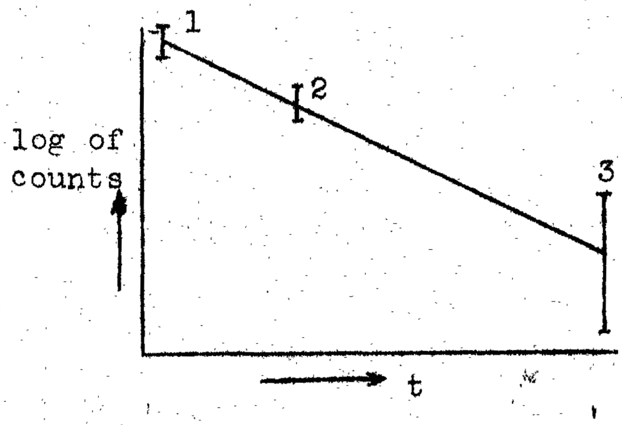
A closely connected question is that of the coincidence corrections to be applied to observed counting rates. Let us assume that a counter takes the time τ to recover. This means that after a count, the counter remains insensitive for a period τ. If we observe a counting rate m_{exp}, the true counting rate m_{corr} is then given by

$$m_{corr} = m_{exp}\, e^{m_{corr}\tau}$$

the exponential factor representing the correction due to the fact that counts are missed if they follow each other at intervals smaller than τ. Assuming that the time τ is short, so that the probability of missing any count due to the above effect is small, one finds that the corrected counting rate m_{corr} is expressed in terms of the experimental counting rate m_{exp} by

$$m_{corr} = m_{exp}(1 + m_{exp}\tau)$$

More involved problems arise if the substance decays appreciably while the experiment is performed. One practical problem is to find which periods of measurement are best suited for the determination of the half-life of the substance. One might guess that it will be best to let the substance decay for many half-lives and to take advantage of the great change in counting rate which has occurred in the meantime. However, if one waits too long, there is too great a loss of accuracy due to the reduced counting rate and the corresponding greater influence of fluctuations. In the following figure the logarithm of the counting rate is plotted against the time.



According to the radioactive decay law this plot should be a straight line. If one determines its value at points 1 and 3 one might expect a greater accuracy in the slope. But this is true only if the measurement at 3 does not become too inaccurate as has been indicated in the figure by the vertical line. It is found actually that it is best to choose the two points 1 and 2 at which counting rates are determined one, or one and a half, half-lives apart.

The fluctuations which we have discussed are a consequence of the radioactive decay law. It is therefore of fundamental interest to see whether those fluctuations conform to the predictions. This was found to be the case whenever measurements were performed with the necessary precautions. One mistake which frequently has led to spurious deviations from the predicted law of fluctuations is to neglect the coincidence correction of the counter.

III. PASSAGE OF PARTICLES THROUGH MATTER

-particles

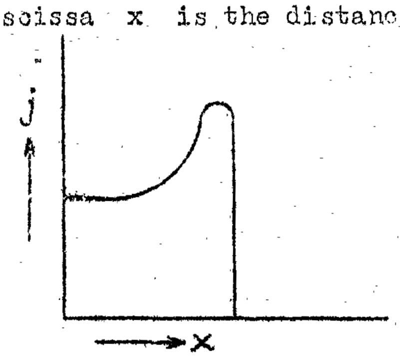
Measuring deflections of α-particles in electric and magnetic fields Rutherford had established that α-particles are helium nuclei.

The characteristic property of the α-rays in which we are here interested, is their definite range. This means that α-particles of a given velocity (for instance α-particles emitted by a thin foil of polonium) will traverse a given distance in air before being stopped. This can be demonstrated in the Wilson cloud chamber.

The Wilson cloud chamber is a chamber containing saturated vapor. Sudden expansion of the chamber causes supersaturation in consequence of adiabatic cooling. The droplets will then condense around ions formed by the passage of the α-particle thus marking the track of the α-rays. The tracks seen in the Wilson chamber are of quite uniform length.

The stopping of α-particles is due to collisions with electrons. Since the mass of the electron is about 7,400 times smaller than that of the α-particle, the α-particle is not deflected by such a collision appreciably and its path remains visibly straight. However, the energy lost in such collisions accumulates. It occurs sometimes that an α-particle collides with a nucleus and suffers a strong deflection. These events can be easily seen in the Wilson chamber.

The following figure shows the density of ionization j along the path of an α-particle. The abscissa x is the distance the α-particle has



travelled in air. The above curve is called the Bragg curve. Its last and best determination is due to Livingston and Holloway[2]. Close to the end of the range, the Bragg curve attains its maximum, making about 6,000 ion paths per mm of air. At the end of the range the ionization declines sharply to zero.

Ionization in air is accompanied by loss of energy. Each ion path corresponds to an energy loss of about 32 eV. This figure is fairly independent of the velocity of the α-particle. Thus the Bragg curve gives not only a measure of the ionization density but also of the energy lost per mm of path.

It is of great interest to find out the energy loss of α-particles in other materials than air. Thus it is important to know the energy lost by the α's when entering a chamber or counter through a thin foil. The energy loss or stopping power in various substances is best measured by giving the value of the energy lost if the particle traverses a foil whose weight per square centimeter is one milligram. If comparing foils of equal weight, materials with more highly charged nuclei have smaller stopping powers.

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2) Phys. Rev. 54, 18 (1938).

October 14, 1943

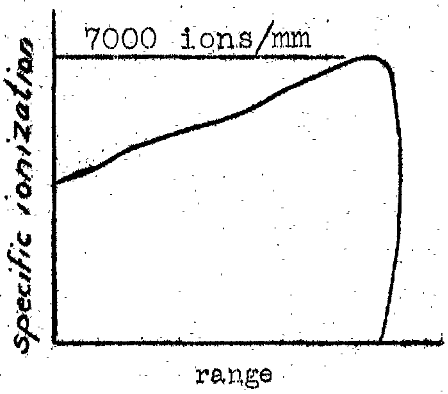
LECTURE SERIES ON NUCLEAR PHYSICS

Second Series: Radioactivity Lecturer: E. Segre

LECTURE X: α-PARTICLES AND THEIR INTERACTION WITH MATTER

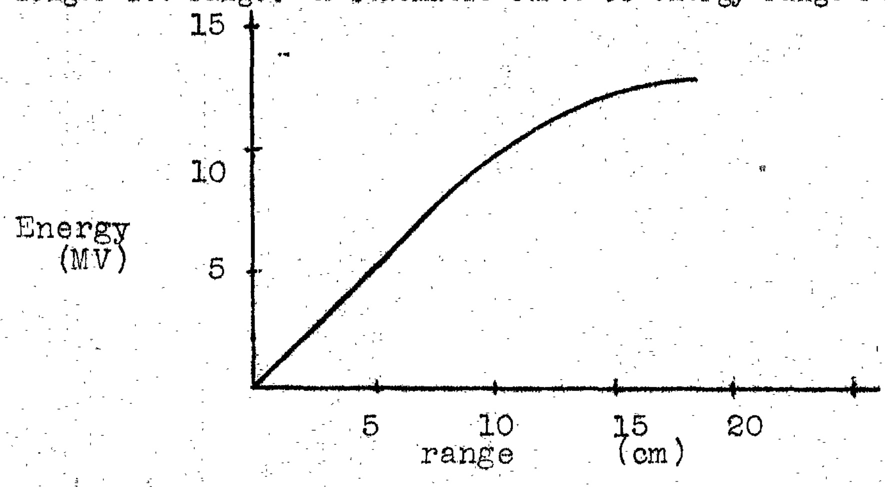
In cloud chamber photographs of an α-emitting source, it is evident that the tracks exhibit a more or less characteristic length, known as the range (v. Rasetti, p. 303). In the main, the energy of the α-particle is dissipated in collisions with electrons; the path of the α-particle is not deflected by these many encounters because of the relatively great mass of the α-particle compared to that of the electron. Upon close inspection of the tracks, one may see the faint tracks of the projected electrons, all having their origins in the heavy track of the α-particle. Occasionally an α-track shows a fork-like structure; here the α-particle has suffered a nuclear collision and may be appreciably deflected, the path of the struck particle being the other branch of the fork. A measurement of the angles and ranges (energy) of the various colliding bodies shows that the conservation laws of momentum and energy are satisfied. Sometimes, one finds that kinetic energy is apparently not conserved. In these cases an inelastic collision or nuclear reaction has taken place.

The many collisions of an α-particle with electrons result in ion production along its path. The number of ions per unit length is called the specific ionization. A curve (Bragg curve) of specific ionization vs range is shown in the following figure.



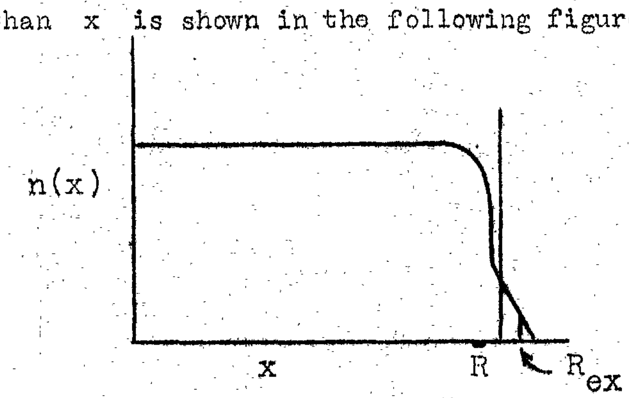
The specific ionization increases with decreasing velocity of the α-particle, attains a maximum near the end of the path and then drops to zero. (See for a quantitative graph, Livingston and Holloway, Phys. Rev. 54 18, 1938.)

It is evident that the greater the initial energy of the α-particle, the greater the number of encounters necessary to dissipate its energy, hence the longer its range. A schematic curve of energy-range relation is given below.



As an example, α-particles from polonium have an energy of 5.298 MeV and a range of 3.842 cm.

Straggling: If the ranges of an initially homogeneous beam of α-particles are measured, one finds values distributed about a certain value, the mean range $\bar{R}$, with a deviation of one or two percent. A plot of the number, $n(x)$, which have a range greater than x is shown in the following figure.



The value of x at which $n(x) = 1/2\ n(0)$ is equal to $\bar{R}$. The intersection with the x-axis of the tangent to the curve at $\bar{R}$ is the extrapolated range.

The distribtuion of ranges (the phenomenon of straggling) may be explained by the statistical fluctuation of the energy loss per collision, and also because the charge of the α-particle varies several thousand times, i.e., as He^{++}, He^{+}, He, along its path. These changes, however, occur almost entirely in the last few millimeters of range.

The stopping power of a substance ~~if~~ is the space rate of energy loss of the impinging particles as they traverse the substance, i.e., if F = stopping power, T = kinetic energy of particle, x = space coordinate,

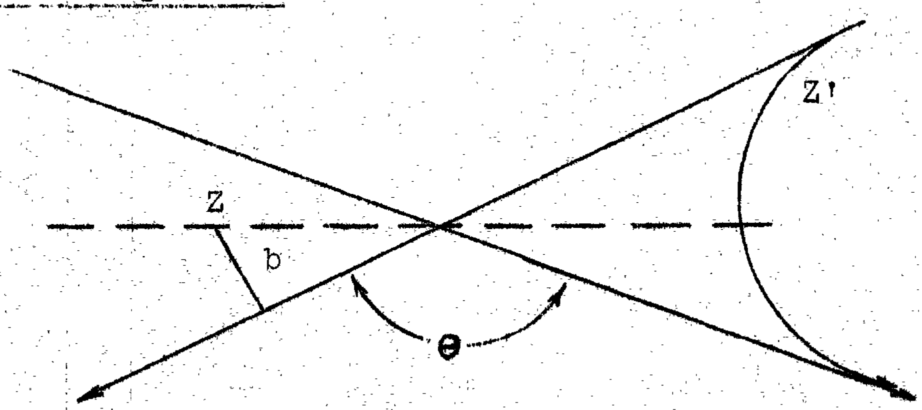
$$F = -dT/dx.$$

Consequently the range R is given by

$$R = \int_0^R dx = -\int_{T_0}^{0} dT/F = \int_0^{T_0} dT/F$$

The relative stopping power of a substance is defined as $\frac{\text{range in air}}{\text{range in substance}}$, approximately independent of energy. Finally, the mass stopping power is the stopping power divided by the density of the substance. The mass stopping power is inversely proportional to the square root of the atomic weight of the substance. E.g., 1 cm of air at 15° and 760 has the same stopping power as 1.62 mg/cm² of Al, 2.26 mg/cm² Cu, or 3.96 mg/cm² Au.

<u>Rutherford scattering formula:</u>



An impinging particle of mass m and initial velocity v_0, and charge Z' (in present case, an α-particle Z' = Ze) approaches a nucleus of charge Ze and, for simplicity, suppose its mass to be infinite. Assuming an inverse square law of force, it is known that the classical motion is a hyperbola with the scatterer at one focus. The change from the initial direction of approach to the final direction of the particle is the scattering angle θ. The hypothetical distance of closest approach if the particle were not deflected is the so-called impact parameter, b of the above figure.

From Newtonian mechanics it turns out that

$$\tan \theta/2 = (ZZ'e^2)/(mv_o{}^2b).$$

Experimentally a beam of particles impinges on a target; the number of particles scattered at an angle θ per unit solid angle $n(\theta)$ is of interest. Let n_o be the number of incident particles per unit volume of the beam, and assume at first only one scatterer. The number of particles scattered by an angle between θ and θ $d\theta$ will be equal to the number having an impact parameter between b and b db, related by the above formula. Hence per unit solid angle

$$n(\theta) = (n\ 2\ \ dbd)/(2\ \ \sin\theta\ d\theta)$$

Using (1), we can eliminate bdb and obtain:

$$n(\theta) = n_o\ \frac{Ze^2}{mv_o^2}\ {}^2\ \frac{1}{\sin^4\theta/2}, \qquad \text{(taking } Z' = 2).$$

If there are N scatterers per unit surface of target, one then simply introduces the factor N.

LA 24 (11)

October 26, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

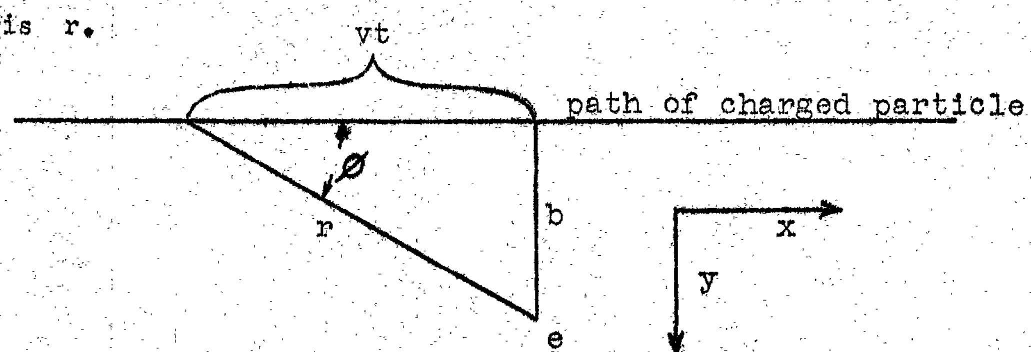
Second Series: Radioactivity Lecturer: F. Bloch

LECTURE XI: THE THEORY OF STOPPING POWER

We shall calculate the energy which charged particles lose when passing through matter. This energy loss determines the range of charged particles (e.g., α-particles or protons). The energy loss is also closely connected with ionization caused by the charged particle in the substance through which it passes. We shall first discuss the theory of stopping power as given by Bohr. His derivation uses only the simplest concepts of classical mechanics.

We consider a particle with charge Ze and velocity v. It interacts with an atom. More particularly we shall examine the interaction with one electron of charge e and mass m within that atom. The electron shall be bound to an equilibrium position with elastic forces and shall have the vibrational frequency ω. The electron was originally at rest. We shall calculate the amount of energy ΔT that the electron obtains during the passage of the α-particle. The charged particle must then have lost the same amount of energy, that is, it has changed its energy by $-\Delta T$.

In the figure, the position of the electron is shown by the point designated as e. The path of the charged particle is represented by the straight line. The distance between the electron and the charged particle at a certain time t is r.



The minimum value of this distance, that is, the distance of closest approach, is equal to b, this distance is called the collision parameter. The angle included by the path of the charged particle and by r is called ϕ. We shall use a coordinate system whose x direction coincides with the path of the α-particle and whose y direction is perpendicular to it. At the point of closest approach of the charged particle to the electron we set $x = 0$. We furthermore assume that for this point also $t = 0$. Then at all other times we have $x = vt$.

We shall solve the problem of the energy transfer to the electron in an approximate but very simple way. The approximation to be used requires: 1) that the electron may be considered essentially as free during the time of collision; and 2) that the displacement of the electron during the collision shall be small compared to the distance of closest approach b.

In order to see whether the first assumption is valid we shall consider the y-component of the force of interaction

$$F = \frac{Ze^2}{r^2}$$

This y-component is

$$F_y = \frac{Ze^2}{r^2} \sin \phi$$

The value of this component is zero at $t = -\infty$ and $t = +\infty$. It has a maximum at $t = 0$. The width of this maximum (that is the time during which F_y is more than one half its maximum value) we call the collision time τ. The condition that the electron may be considered as free is fulfilled if τ is small compared to the period of vibration $1/\omega$. Indeed if this is the case the electron moves during the collision through a distance which is small compared to its vibrational amplitude, i.e., the electron remains close to its equilibrium position. But near the equilibrium position the elastic forces are small, they may be neglected and the electron may be considered as free. We shall discuss the validity of our second assumption later.

In order to calculate ΔT we shall first compute the momentum given to the electron by the charged particle. According to our second assumption, the electron has moved a small distance compared to b and the force will be practically the same as though the electron has not moved at all. Under these conditions one sees that the net change in the x-component of the momentum of the electron, Δp_x is equal to zero. Indeed the momentum in the x direction that the electron has obtained while the charged particle was approaching, will be cancelled by a momentum change in the opposite direction that occurs while the charged particle recedes. Thus the only net momentum change is in the y-direction and after the passage of the charged particle the electron is left vibrating in that direction. This momentum change p_y is the time integral of the y-component of the force of interaction.

$$\Delta p_y = \int_{-\infty}^{\infty} F_y \, dt = \int_0^{\pi} \frac{Ze^2}{r^2} \sin \phi \frac{dt}{d\phi} d\phi$$

Substituting

$$r = b/\sin \phi ; \frac{dt}{d\phi} = \frac{b}{v} \times \frac{1}{\sin^2 \phi} .$$

one finds

$$\Delta p_y = \frac{Ze^2}{bv} \int_0^{\pi} \sin \phi \, d \phi = \frac{2Ze^2}{bv}$$

The corresponding change in kinetic energy is

$$\Delta T = \frac{1}{2m} \quad (\Delta p_x^2 + \Delta p_y^2) = \frac{2Z^2e^4}{mb^2v^2}$$

This formula holds as long as $b_{min} < b < b_{max}$. If b is outside this region, the assumptions made above are not valid.

We shall calculate now the enrgy lost by the particle when passing through a distance Δx. Let us assume that there are N electrons per centimeter cubed. In a cylindrical shell of length Δx, inner and outer radii b and

$b + db$ there will be $2\pi b db \Delta x N$ electrons. The energy transferred to these electrons will be

$$\Delta T = N\Delta x \, \frac{4\pi Z^2 e^4}{M \, v^2} \int_{b_{min}}^{b_{max}} \frac{db}{b} = N\Delta x \, \frac{4}{m} \, \frac{Z^2 e^4}{v^2} \, \log \frac{b_{max}}{b_{min}}$$

From this equation the energy loss per centimeter $\Delta T/\Delta x$ may be obtained as soon as the values of b_{min} and b_{max} are determined.

Discussion of b_{min}

For $b = 0$ our approximate formulae would give $\Delta T = \infty$ which is incorrect. Actually the maximum velocity a free electron may obtain in a collision is 2v and therefore the maximum energy loss is

$$\Delta T_{max} = \frac{m}{2}(2v)^2 \;?$$

$$\Delta T_{max} = \frac{m}{2}(2v)^2 = 2mv^2$$

If we choose b in such a manner that the calculated energy loss shall be equal to this maximal energy loss, then this value of b will mark the limit beyond which the approximate value of the energy loss can no longer be used. At this value of b, the assumption that the electron has been displaced during the collision through a distance small compared to b is no longer valid. This value of b we shall choose as b_{min} we obtain

$$\frac{m}{2}(2v)^2 = \frac{2Z^2e^4}{m \, b_{min}^2 v^2}$$

$$\frac{m}{2}(2v)^2 = \frac{2Z^2e^4}{m(b_{min})^2 v^2}$$

$$\cancel{2} mv^2 = \frac{\cancel{2} Z^2 e^4}{m(b_{min})^2 v^2}$$

or

$$b_{min} = \frac{Ze^2}{mv^2}$$

$$b_{min} = \frac{Ze^2}{mv^2}$$

A treatment which is rigorous for small values of b uses the Rutherford formula for the scattering of the charged particle by the electron. This treatment gives

$$\Delta T = N\Delta x \, \frac{4}{M} \, \frac{Z^2 e^4}{v^2} \int_0^{b_{max}} \frac{b\,db}{b^2 + (2Ze^2/mv^2)^2}$$

In the case that

$$b \gg \frac{Ze^2}{mv^2}$$

the second term in the denominator can be neglected and the integral reduces to the simpler form which has been used above. At small values of b the integral goes to zero. This gives a qualitative justification for using the simpler integral and for cutting off the integration at

$$b = b_{min} = \frac{Ze^2}{mv^2}$$

Discussion of b_{max}

For too large values of the collision parameter the electron can no longer be considered as free. In fact, the collision time is approximately $\tau = b/v$ and this time will become greater than the vibrational period of the electron $1/\omega$ when b becomes sufficiently large. For large values of b, that is, when $b/v \gg 1/\omega$ the variation of the force of interaction between charged particle and electron is slow compared to the vibrational frequency. In this case the interaction between the charged particle and the electron may be treated by the so-called adiabatic approximation. In this approximation, the electron is displaced when the charged particle approaches but when it recedes the electron is eased back into its original position without it taking any energy. It is therefore justified to extend our integral to the point where $b/v = 1/\omega$ and to neglect contributions from greater b values. Thus we have

$$b_{max} = v/\omega$$

Substituting the values of b_{max} and b_{min} into the energy loss formula obtained above, one finds

$$\frac{dT}{dx} = N \frac{4\pi Z^2 e^4}{mv^2} \log \frac{mV^3}{Ze^2\omega}$$

As has been pointed out this formula is only approximate. The calculation has been carried out by Bohr rigorously and he obtains

$$\frac{dT}{dx} = N \frac{4\pi Z^2 e^4}{mv^2} \log \frac{1.123\ mV^3}{Ze^2\omega}$$

The condition that is necessary for the validity of this formula is that there be a range of b-values in which $b \gg b_{min}$ and $b \ll b_{max}$. It is therefore necessary that $b_{min} \ll b_{max}$. This means that

$$\frac{Ze^2}{mv^2} \ll \frac{v}{\omega}$$

or

$$\frac{Ze^2\omega}{mv^3} \ll 1$$

One sees that the Bohr formula is valid only if v is sufficiently large. As an example we may consider the α-particles of RaO'. Their velocity is $v = 1.92 \times 10^9$ cm/sec. For ω we may use the frequency of the outermost electrons that is an optical frequency of $\omega = 10^{16}$ sec^{-1}. This gives

$$\frac{Ze^2\omega}{mv^3} = 3.10^{-4}$$

We see therefore that in this case Bohr's formula is valid.

October 28, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Second Series: Radiactivity Lecturer: F. Bloch

LECTURE XII: THE THEORY OF STOPPING POWER (CONTINUED)

In deriving the formula for the stopping power, it has been assumed that electrons in atoms move like harmonic oscillators. Actually, they are not subject to harmonic forces, but rather to Coulomb forces. Bohr had assumed harmonic oscillations because in that case alone does classical theory give rise to sharp lines as are observed in spectra.

It can be shown in quantum treatment that interaction of light with atoms may be described as the interaction of light with a series of virtual oscillators. This means that instead of an assembly of atoms we might actually consider an assembly of harmonic oscillators whose frequencies are equal to the absorption frequencies of the atoms. If we have n atoms per cubic centimeter we must replace these atoms with nf_i oscillators with the absorption frequency ω_i. The quantity f_i is called the oscillator strength and satisfies the sum rule

$$\Sigma f_i = Z_o = \text{total number of electrons in an atom.}$$

This means that the total number of oscillators of various frequencies is equal to the total number of electrons actually present in the body.

If a charged particle moves past an atom, it acts on the electrons of the atom by virtue of its electric field. A light wave too would act on an atom by its electric field and it is therefore justified to replace the atoms by oscillators and then use the formula for the energy loss which we had derived. In this way we obtain the following formula

$$-\frac{dT}{dx} = \frac{4\pi e^4 Z^2}{mv^2}\, n \sum_i \int_{b^i_{min}}^{b^i_{max}} f_i \frac{db}{b}$$

Here n denotes the number of atoms per cubic centimeter. For the upper limit we may use the same expression as was derived in classical theory, namely,

$$b^i_{max} = v/\omega_i$$

The lower limit

$$b^i_{min} = Ze^2/mv^2$$

however, must be changed. This classical lower limit has the following significance. If we consider the kinetic energy of the electron in the frame of reference where the charged particle is at rest, then b^i_{min} is the distance at which the potential energy between the electron and the charged particle becomes equal to that kinetic energy. Now in quantum mechanics a closer approach between electron and charged particle than the de Broglie wave length is meaningless and we have to use therefore for the lower limit

$$b^i_{min}\text{ (quantum)} = \hbar/mv.$$

This is indeed so if the lower limit in quantum theory has a higher value than the previously derived classical lower limit, because at any approach closer than the quantum lower-limit defraction phenomena will occur and make an effective approach impossible. If however, the classical lower-limit should turn out to be greater, then down to that lower limit, classical theory is applicable and the classical lower-limit should be used. Thus, the classical or quantum lower-limit must be used according to whether the ratio

$$\frac{b_{min}\text{ (classical)}}{b_{min}\text{ (quantum)}} = \frac{Ze^2}{\hbar v}$$

is big or small compared to unity. Assuming $Ze^2/\hbar v \ll 1$, one obtains for the energy loss

$$\frac{dT}{dx} = \frac{4\pi e^4 Z^2}{mv^2} \sum_i f_i \log \frac{2mv^2}{\hbar\omega_i}$$

The formula actually given contains the numerical factor 2 under the logarithm which factor does not follow from our argument but must be obtained by a more detailed calculation.

The stopping power just given is valid if the following three conditions are fulfilled:

(a) $v/\omega_i \gg r_0$

(b) $Ze^2/\hbar v \ll 1$

(c) $mv^2 \gg \hbar\omega_i$

Condition (a) means that at the upper limit where we have made use of the frequency of the electron, the distance of the charged particle of the atom should be great compared to the atomic radius r_0. This condition is necessary in order that the electric field of the charged particle should be homogeneous over the atom whenever the oscillator treatment of the electron has been made use of. Unless this condition is satisfied, the analogy to the interaction with light and the treatment by virtual oscillators breaks down. Condition (b) means that the quantum lower-limit is greater than the classical lower-limit and that therefore the quantum lower-limit should be used. Finally, condition (c) expresses the requirement that the upper limit of b must be considerably greater than the lower limit of b and that therefore there shall be a region in which our approximations are valid.

Conditions (a) and (c) both mean numerically that the velocity of the incoming charged particle must be great compared to the velocity v_{el} of the electron with which the charged particle interacts. For condition (a) this is seen readily because $r_0\omega$ is equal to v_{el}. In condition (c) one may replace $\hbar\omega_i$ (that is, the excitation energy of the electron) by twice the kinetic energy of the electron which is a quantity of the same order of magnitude. Then (c) reduces to

$$mv^2 \gg mv_{el}^2$$

which again means that $v \gg v_{el}$. Condition (b) is also satisfied if $v \gg v_{el}$, provided that Z is not great compared to 1. This is so because $e^2/\hbar v_{el}$ is about equal to 1 for the outermost electrons and is smaller than 1 for all other electrons.

If v and v_{el} become comparable, our treatment is no longer valid and an exact treatment for this case has not yet been carried out. The theoretical difficulties are paralleled by a complicated behavior of the charged particles at relatively slow velocities. If their velocity becomes equal to the velocity of the outermost electrons in the slowing down medium, then the charged particles may pull out electrons from the slowing down medium, attach them to themselves, and thus reduce their effective charges. In later collisions these charges may again be lost. While the charged particle had its charge reduced, its ionizing power and energy loss will also be smaller. Since various charged particles will spend various times in the state of reduced charge, the actual range of the individual particles will differ. This difference of range is called straggling.

If an α-particle has an energy of 72 kV, then its velocity is the same as that of an electron with 10 eV. This is the average of the outermost electrons. Thus, we must expect that our formula of stopping power for α-particles becomes inapplicable under 72 kV. Actually, serious deviations from the formula start at the somewhat higher energy of 200 kV. The corresponding limit for deuterons is at 100 kV and for protons at 50 kV.

From the formula of the stopping power an estimate of the range of the particles can be obtained. For this purpose, we replace the energy loss dT/dx by the change in kinetic energy of the particle $-dE_{kin}/dx$. The formula for the stopping power then gives the proportionality

$$-\frac{dE_{kin}}{dx} \sim \frac{1}{E_{kin}} \log \frac{E_{kin}}{\text{const}}$$

Forgetting about the logarithmic factor, one obtains

$$E_{kin} \frac{dE_{kin}}{dx} = \frac{1}{2} \frac{d(E_{kin}^2)}{dx} = -1$$

This formula shows that the quantity E_{kin}^2 changes at a constant rate along the path of the particle and that therefore the range should be proportional to the square of the original kinetic energy. Actually, the range varies with a somewhat lower power of the original energy. This is due to the logarithmic factor in the stopping power formula.

A quantitative evaluation of the stopping power formula requires knowledge of all absorption frequencies and all oscillator strengths. These quantities can be given in an analytic form for the hydrogen atom and in this case the required calculations have been carried out.

For heavy atoms one may use a crudely simplified statistical model. This model considers the electrons as a gas, the kinetic energy of the electrons being balanced by the average electrostatic potential within the atom. One may find the oscillation frequency of this gas by hydrodynamic calculation. One then may use that frequency instead of ω_i and use the total number of electrons in an atom Z_0 instead of the oscillator strength f_i.

The vibrational frequency in the statistical model will be given by the "sound velocity" divided by the atomic radius. The atomic radius can be shown to vary as $Z_0^{-1/3}$*. The sound velocity is of the same order of magnitude as the average velocity of the electrons and that is proportional to $Z_0^{1/3}$. Thus we obtain

$$\frac{dT}{dx} = \frac{4\pi e^4 Z^2}{mv^2} Z_0 \log \frac{2mv^2}{kZ_0 \quad R}$$

In this formula, R is the so-called Rydberg frequency, that is $\hbar$R is the

* This means that heavier atoms have smaller atomic radii which is in contradiction to the experience about atomic radii as customarily defined. However, in the customary definition the atomic radius is essentially given by the orbits of the few outermost electrons whereas in our present discussion the radius is the average distance of electrons from the nucleus, the average being taken over all electrons.

ionization energy of the hydrogen atom, and k is a numerical factor. Instead of calculating it, one may adjust it in such a way as to give agreement between the formula and the experience for one heavy atom, for instance for gold. Then the formula describes the dependence of the stopping power on the atomic number in a satisfactory way.

The formula for the stopping power of electrons is the same as the formula for other charged particles with the exception that under the logarithm the factor 2 does not appear:

$$\frac{dT}{dx} = \frac{4\pi e^4 Z^2}{mv^2} \sum_i f_i \frac{mv^2}{\hbar\omega_i}$$

This is due to the fact that in determining the lower limit of b the de Broglie wave length must be used which corresponds to the velocity of the atomic electron in a frame of reference where the center of gravity of that electron and the oncoming charged particle is at rest. This velocity is practically equal to the velocity of the oncoming particle, as long as that oncoming particle is heavy, but if the particle is an electron itself, the velocity in the center of gravity system is one half the velocity of the oncoming electron. In the case of fast β-particles and of charged cosmic-ray particles the motion of the oncoming particle must be treated according to relativistic mechanics. This will have two consequences: 1) the region in which the electric field acts is contracted by the Lorentz factor $\sqrt{1 - v^2/c^2}$. The time during which the field acts is shortened by the same factor; 2) the strength of the electric field is increased by the reciprocal of the Lorentz factor. Therefore, the momentum and energy transferred to an electron during the collision remain the same as in unrelativistic theory. However, the limits for the collision parameter b change. The upper limit is increased by the reciprocal of the Lorentz factor because, due to the Lorentz contraction, we can go to greater distances before the collision time becomes as long as the period of the atomic electron. In the lower limit the De Broglie wave length must again be used which is given by

$$\frac{\hbar}{\text{momentum}} = \frac{\hbar}{mv}\sqrt{1 - v^2/c^2}$$

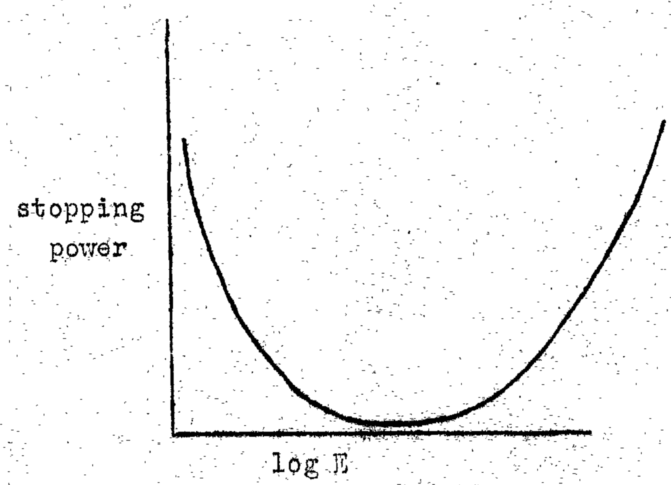
and where m is the restmass of the electron. We see however that this limit is no longer

$$\frac{\hbar}{mv}$$

but is shorter than this length by the Lorentz factor. Therefore the square of the Lorentz factor will appear in the denominator of the logarithm in the stopping power formula,

$$\frac{dT}{dx} = \frac{4\pi e^4 Z^2}{mv^2} \; n \sum_i f_i \log\left[\frac{mv^2}{\hbar\omega_i(1 - v^2/c^2)}\right]$$

It should be remarked that the first factor of this formula is no longer inversely proportional to the energy of the charged particle. Indeed for high energies mv^2 in the denominator approaches the constant value mv^2; thus this factor ceases to decrease while the logarithmic factor increases due to the expression $1 - v^2/c^2$ in its denominator. Thus the stopping power has a minimum as shown in the figure.



For electrons this minimum is obtained at about 1 MV.

The stopping power formula for electrons is difficult to compare with experiments due to variation in the ranges of individual electrons that is due to straggling. In the case of electrons, straggling is caused however by entirely different reasons than have been discussed in the case of heavy particles. These reasons are 1) that the electron may lose in one collision a very considerable part of its energy with a rather high probability and therefore a statistical treatment of the energy loss is less justified than in the case of heavy particles; and 2) electrons may be deflected appreciably in collisions with other electrons and nuclei and so the path of the electrons is much less straight than the path of the heavy particles, and, for example, in the perpendicular transversal of a plate does not correspond to the thickness of this plate.

LA-24 (13)

November 2, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Second Series: Radioactivity Lecturer: E. Teller

LECTURE XIII: INTERACTION OF CHARGED PARTICLES WITH MATTER

In the previous lecture it was shown that, apart from other factors, the (space) rate of energy loss of a moving charged particle (α-particle was the example) was inversely proportional to the square of the velocity v of the particle. This factor arises because the momentum transferred to an electron is proportional to the "time of collision" which in turn is inversely proportional to the particle velocity. Hence a particle of initial energy E loses energy at a rate $\sim 1/E$ and its range should therefore be $\sim E^2$. Experimentally, however, the range is more nearly $\sim E^{3/2}$. The discrepancy may be removed by a more critical examination of the energy loss equation. It will be recalled that the incident particle will transfer energy to the atomic electrons if its velocity is greater than the electron velocities. Approximately, the number of electrons in an atom which have velocities smaller than v, and can therefore receive energy from the particle, is simply proportional to v (provided v is in the range of electron velocities). Consequently, the rate of energy loss is now $\sim 1/v \sim E^{-1/2}$ and the range is proportional to $E/E^{-1/2} = E^{3/2}$.

It is of interest to note that the energy loss per unit length of path depends on the speed and charge of the incident particle. Particles of the same speed lose energy at a rate proportional to square of their respective Z's, so that a 4 MV α-particle (Z = 2) has the same range as a 1 MV proton (Z = 1).

The energy which is taken from the incident particle may be used to eject an electron from atom (ionization), or may only produce electronic excitation with subsequent emission of a light quantum (e.g., scintillations observed in detection of α-particles). Another possible process, and of considerable importance in connection with biological effects of radiation, is molecular excitation with subsequent dissociation into atoms or ions. Taking into account these various processes together with secondary effects, one finds that the average energy per ion pair is roughly 30 eV. An approximate measure of the biological effects of radiation on a part of the body is the amount of energy absorbed by the particular region. There are, of course, qualitative differences, because the density of ions is quite different along paths of various impinging particles, so that recombination processes may be of another character in the case of, say, a heavily ionized α-track as compared to a weak β-track.

Ionization by heavily charged particles.

We have seen that the Bragg curve of an α-particle, a measure of the specific ionization along the path, increases slowly to a maximum and then drops sharply to zero. The corresponding curve for fission particles shows only a decrease in specific ionization along its path. What is the explanation for the difference? The initial velocity of a fission particle is approximately 10^9 cm/sec. One can estimate its initial charge because the fission fragment will shed all the electrons having velocities smaller than its own. As it proceeds along its pa th, the particle will pick up electrons, as a result the specific ionization will tend to decrease. At approximately 1 MV, the particle becomes neutral; its path as a charged particle is about five to ten times longer than the distance it travels as a neutral particle. The range is about 2 cm. in air.

Stopping of Electrons.

It is difficult to ascribe a range to electrons traveling through matter because of the excessive straggling. In contrast to the essentially straight path of an α-particle, an electron may be considerably deflected in a collision, so that it will have a more or less random path. Secondly, the energy spectrum of β-particles emitted from a nucleus is not monochromatic (as in the case of α's) but has a considerable spread. The combined effect gives a range distribution which imitates an exponential absorption curve.

There are several processes by which the energy of a fast moving electron may be dissipated. When an electron passes through the electric field of a nucleus, it is accelerated. This acceleration gives rise to radiative energy losses (bremsstrahlung). They are appreciable for electron energies of several Mev. The electron-electron radiative losses are quite small inasmuch as there can be no dipole, but only quadripole, radiation.

The electron suffers energy losses in its collisions with electrons. An approximate formula has been given for this rate of energy loss, and we have seen it depends on the square of the atomic number of the nuclei. In these collisions, all of the atomic electrons participate and ionization in the traversed material is produced. The ratio of ionization loss to radiative loss is given roughly by

$$\frac{\text{Rate of ionization loss}}{\text{Rate of radiative loss}} = \frac{Z\ (\text{mev})}{800}$$

where Z is the atomic number of the material, E is the kinetic energy in million electron volts.

To give an idea of how much shielding is necessary to stop β-particles, one may give an example. For aluminum 100 mg/cm^2 will stop 100 kv betas, and 1 g/cm^2 will be adequate for 3MV betas. It is approximately true that the stopping power per unit mass is the same for all the elements.

<u>Reflection of β particles.</u>

Coefficients of reflection between 1/10 and 1/2 are quite usual. Heavy nuclei by virtue of their charge are more effective in producing large angle scattering than light nuclei. Hence pb reflects β particles much better than, say, paraffin.

LA-24 (14)

November 4, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Second Series: Radioactivity Lecturer: E. Teller

LECTURE XIV: INTERACTION OF RADIATIONS WITH MATTER

An examination of a fission particle track reveals branching tracks that originate from the main track. These secondary tracks are short and are especially numerous at the end of the principal track, the resulting appearance being life a tuft of an arrow. To interpret this behavior, let us consider the ratio of energy transfer to a nucleus and to its surrounding electrons. If Z_1, Z_2 are the atomic numbers of stopping and fragment nuclei respectively, the rate of energy transfer to a nucleus of mass M is proportional to $\frac{Z_1^2 Z_2^2}{M}$. In the electronic collision, it is the effective charge Z_2' of the fission fragment that is considered in the rate of energy loss; hence for this type of collision, the rate is proportional to $Z_1(Z_2')^2/m^o$, since there are Z_1 electrons per nucleus. The ratio $Z_1(m/M)\ (Z_2/Z_2')^2$ will increase considerably near the end of the fission particle track because the effective charge decreases; hence the relative rate at which energy is lost by nuclear collisions increases markedly near the end of the track. Throughout the track the ratio is greater than it is in case of α-particles. Hence branchings due to nuclear collisions are more frequent.

Interaction of γ-rays with matter.

The main source of energy loss of γ-rays is interaction with electrons. However, interaction with nuclei has been observed. One example is the nuclear photoelectric disintegration. Two instances may be given:

$$D + \gamma \longrightarrow n + p$$

$$Be + \gamma \longrightarrow n + 2\alpha$$

In particular, the first reaction is important because it gives information concerning the binding energies and forces between elementary particles. Cross-sections for this type of reactions are relatively small, being of the order of 10^{-27} cm^2 or smaller. The two reactions cited above require relatively low γ-energies, i.e. about 2 Mev. Most nuclear photoeffects require 5-8 Mev γ-rays.

The interaction of γ-rays with matter are in the main interactions with electrons. The phenomena may be described as the photo-electric effect, Compton scattering and pair production. We shall see that the various interactions depend differently on nuclear charge and the energy regions in which each effect becomes important is more or less well defined.

Photoelectric effect.

A bound electron absorbs energy and leaves the atom. Most of the energy of the γ-quantum is given to the electron, most of the momentum is given to the residual positive ion. Each shell of electrons around a nucleus will contribute whose energy is smaller than the energy of the γ-quantum. Among these the shells whose ionization energy is closest to the energy of the γ-ray makes the greatest contribution. γ-ray energies are usually greater than the ionization energy of the K electrons so that the innermost shells have the largest effect.

For sufficiently energetic γ-rays, the photoelectric absorption coefficient goes as $Z^5/E_\gamma^{7/2}$ where Z is the atomic number of the absorbing nucleus and E_γ the γ-ray energy. This formula indicates a very great difference between the respective coefficients for Pb and C. There is actually a great difference in absorption coefficient in the region where photoelectric effect is the main source of absorption. This is the X-ray region.

Compton Effect

The magnitude of the scattering of a photon by an electron can be estimated by classical arguments. In this scattering process, the bound electrons behave as though they were free. If E is the magnitude of the electric vector, m the electronic mass, and $\ddot{x}_e$ the acceleration, then

$$\ddot{x}_e = \frac{eE}{m}$$

It is this acceleration which gives rise to the scattered radiation. From classical theory, we know

$$W = \text{energy emitted per sec per cm}^3 = \frac{Ze^2}{3}\,\frac{\ddot{x}^2}{c^3}\,N$$

Substituting from above for $\ddot{x}$, one gets, for N electrons/cm^3

$$W = \frac{Z}{3}\,\frac{e^4E^2}{m^2c^3}\,N$$

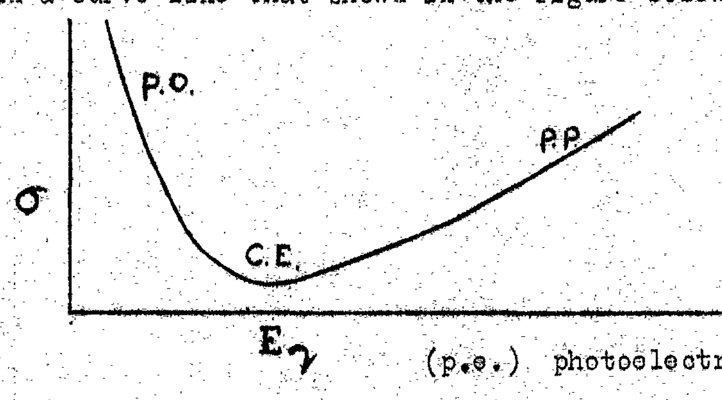
The incident energy density per second is $(E^2/4\pi)C$. Hence the fraction, or scattering coefficient is $(8\pi/3)\,(e^2/mc^2)^2\,N$. Now e^2/mc^2 is the classical electron radius, so approximately the electrons scatter as though they were discs of radius $\approx e^2/mc^2$. E_γ must be less than mc^2 in this approximation. Klein and Nishina have derived a relativistic treatment which holds for E_γ greater than m_ec^2. Then the scattering cross-section falls off approximately as $1/E_\gamma$, while for $E_\gamma \ll mc^2$ the cross-section was independent of E_γ.

In the Compton effect a γ-quantum not only changes the direction of its motion but also loses some energy. This energy loss is particularly significant if originally $E_\gamma > mc^2$. The Compton scattering process was a striking proof of the conservation laws of energy and momentum in a single process, rather than as an average over many processes.

Pair Production.

In this process, a γ-ray disappears, an electron and a positron appear. On the theoretical side, the picture is a continuum of negative energy levels filled with electrons. The highest energy is $-mc^2$. Above this, separated by $2mc^2$, begins another continuum of positive energies. Dirac postulated that all the filled negative energy levels are not detectable. However, when one of these electrons is given an energy in excess of $2mc^2$, it could get into the positive energies and become an observable electron. The "hole" left would then behave as an electron of opposite charge. A short time after the proposal of this theory, this "hole" in the sea of electrons, or what is now called a positron, was found experimentally; it was also found that pair production required at least energies $\geq 2mc^2$ in agreement with theory. The process of pair production requires the presence of a nucleus to satisfy the conservation laws of energy and momentum. Pair production increases slowly with energy, approximately as the logarithm of the energy.

To summarize: One can say that the photoelectric effect is important at low energies and decreases rather rapidly with increasing energy, the Compton scattering then becoming important. At still higher energies, the Compton scattering decreases, but then pair production becomes important, so if one were to plot a graph of the sum of the cross-sections as a function of γ-ray energy one would obtain a curve like that shown in the figure below.



(p.e.) photoelectric effect

(C.E.) Compton effect

(p.p.) pair production

For Pb the minimum occurs at about 2MV; for Cu and Al at about 10 and 25 MV respectively.

November 9, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Second Series: Radioactivity Lecturer: E. Segre

LECTURE XV: THE PROPERTIES OF NUCLEI

The properties of nuclei may be discussed under the following headings: Charge, Mass, Size, Density, Spin and Magnetic Moment, Statistics.

Charge

The most direct method of measuring the charge of nuclei is by means of the Rutherford scattering. However, this method is not very accurate. A more accurate determination uses the Moseley law relating the frequency of the K lines in the X-Ray spectrum to the charge of the nuclei. The elements Hafnium and Rhenium were discovered in this manner. Of course, the most frequent method of determining the nuclear charge is by chemical analysis, since as it has been mentioned, it is the charge of the nucleus that determines the chemical properties of the atom.

Mass of the Nuclei

An accurate determination of the masses is of interest, because it gives information on energies of the nuclear reactions. This is possible because of the connection between nuclear energy and nuclear mass in that the energy is obtained from the mass by multiplying the latter with the square of the light velocity.

Up to 1935, Aston had a monopoly on accurate determinations of nuclear masses by the help of the mass spectrograph. Since then, improved mass spectrographs have been constructed by Bainbridge, Jordan and Mattauch. The present accuracy of the mass determination can be illustrated by the statement that the rest energy of nitrogen can be determined mass spectrographically, to within an uncertainty of 10,000 electron volts. This accuracy compares favorably with that of good optical spectrographs.

The method of mass determination utilizes comparisons between ions of nearly equal masses such as:

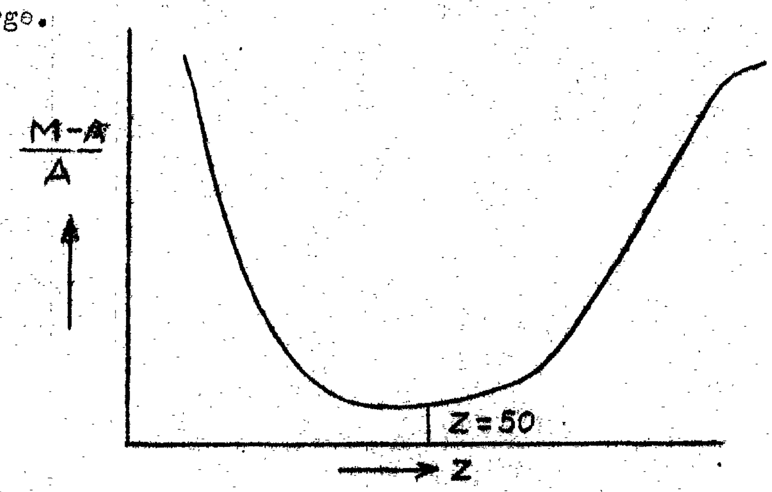
$$(O^{16})^{+} \; (N^{14}H_2)^{+} \; (C^{12}H_4)^{+}$$

Differences of these masses are then measured. For accurate measurements, it is best if the ion concentrations in the discharge are so matched that their intensities are approximately equal.

The scale of measuring masses used in physics is based on the isotope O^{16}. The mass of this isotope is set equal to accurately, 16 units, then one mass unit corresponds to 930 Mev. This quantity might be compared with the mass of the electron which corresponds to .51 Mev.

In chemistry, it is usual to set the atomic weight of the natural isotopic oxygen equal to 16. In this case, the unit of nuclear masses is 1.66035 $\pm .00031 \times 10^{-24}$ grams. In discussing atomic masses, the mass defect is of interest. It is defined as M - A, where M is the mass measured in the physical scale, and A is the integer number closest to M and is called the mass number. A further quantity frequently used is the packing fraction which is defined as $\frac{M - A}{A}$.

In the following figure this packing fraction is plotted against the nuclear charge.



The best figure of this kind is found in the article of Hahn and Mattauch, Physikalische Zeitschrift, 1940. The packing fraction has a minimum at about $Z = 50$.

Another method of determining masses is by balancing reaction energies. For instance, the mass of the neutron has been determined in this manner. This mass, of course, could not be determined in the mass spectrograph because neutrons cannot be deflected in the electromagnetic fields of that apparatus. For the determination of the neutron mass the reaction

$$H_1^2 + h\nu \longrightarrow H_1^1 + n_1^0$$

is used. Knowing the mass spectrographic values of the mass for H_1^2 and H_1^1 which are 2.0147 and 1.00812 and using the minimum energy 2.17 Mev at which the light quantum can cause disintegration, one finds for the mass of the neutron 1.00812. In many other cases, cross-checks are possible by using mass spectrographic values of the masses and data from nuclear reactions.

Size of Nucleus

One may determine the size of atomic nuclei from various reactions, some of these, however, like reactions of capture of slow neutrons, give erratic values. The actual size of the nucleus can be defined as the radius to which specific nuclear forces extend, and is usually obtained by investigating the deviations from the Rutherford scattering or by investigating the potential barrier effects in α-decays and other nuclear reactions involving charged particles. One finds that nuclear radii are given approximately by $1.4 \times 10^{-13} A^{1/3}$. This means that the nuclear volume is proportionate to the nuclear mass.

Nuclear Density

It follows that nuclear densities are approximately the same for different nuclei.

Nuclear Spin and Magnetic Moment

Many atomic nuclei possess angular momenta or spins. These spins were first discovered in a structure of spectral lines which is called hyperfine structure. These structures were first observed at the end of the last century, but their explanation by Pauli and their more detailed study goes back only two decades. The explanation of the hyperfine structure is as follows: The orbital motion of the electrons produces magnetic fields. These interact with the magnetic moment associated with the nuclear spins. The interaction energy modifies the sprectral lines in a different way, according to the orientation of the nuclear spin in the magnetic field of the electronic motion; thus the lines are split. From the number of the components, the spin may be determined, while from the size of the splitting, the nuclear magnetic moment may be calculated.

Data on magnetic spins and magnetic moments have been summarized up to 1939 by Korsching (Zeitschrift fur Physik).

A new method for the determination of the spin and magnetic moments has been developed by Stern and Rabi. This method uses atomic or molecular beams passing through static and oscillating magnetic fields. The method is more direct than the one involving the hyperfine structure. Bloch and Alvarez have determined by this method, the magnetic moment of the neutrons which, of course could not be determined by the use of hyperfine structure. The following table summarizes the results for the neutron, the proton and the deuteron:

	spin	magnetic moment
n	1/2 ?	-1.93
H^1	1/2	2.78
H^2	1	0.855

The spin of the neutron has been assumed as 1/2. This value is probable, but not quite certain. The magnetic moments are measured in units of

$$\frac{eh}{4\pi Mc}$$

where M is the mass of the proton. The minus sign in the magnetic moment of the neutrons indicates that the magnetic moment of this particle has an opposite direction as would be expected from a rotating positive charge.

Statistics

Nuclear statistics show up in spectra in the scattering of nuclear particles and in some cases (like that of ortho-parahydrogen modifications) in physico-chemical properties like specific heat. It will suffice here to say that two kinds of statistics are known - the Bose statistics, which is expected to hold for nuclei containing an even number of particles, and the Fermi statistics which should hold if the number of particles within the nucleus is odd.

It is of historical interest that before the discovery of the neutron, nuclei were supposed to be composed out of protons and electrons; thus N^{14} was supposed to contain 14 protons to make up its mass, and 7 electrons to compensate the 7 excess charges. Thus, one expected this nucleus to contain an odd number of particles and to behave according to Fermi's statistics. Rasetti's spectrographic studies showed that N^{14} behaved according to Bose statistics. According to the newer views, N^{14} consists of 7 protons and 7 neutrons; that is, it contains an even number of particles, and one understands why it obeys the Bose statistics. It follows from the present picture that nuclei with even mass numbers contain an even number of particles and behave according to Bose statistics. One may add that they also have an even spin. On the other hand, nuclei with an odd mass number, contain an odd number of particles and behave according to Fermi's statistics, and have an odd spin (in units $h/2\pi$).

We have just seen that the properties of N^{14} indicate that this nucleus is composed of neutrons and protons rather than of protons and electrons. Originally the latter hypothesis was accepted because of the fact that electrons were observed to come out of nuclei in β-decay processes. This argument, however, cannot be taken very seriously because the same reasoning would lead to the conclusion that excited atoms contain light quanta. In the meantime, it has been observed that in some artificial β-decay processes positrons are emitted. Thus one would have to believe by the same argument that nuclei contain positrons in addition to electrons.

A rather strong argument has been given which shows that neither electrons nor positrons can exist in nuclei. If a particle is confined to a region with linear dimensions Δq then the momentum has an uncertainty $\hbar/\Delta q$, that is, the momentum may have any value from zero up to about this latter value. Now in the nucleus Δq is a few times 10^{-13} cm. From the corresponding momentum uncertainty one may calculate the spread in kinetic energy by the formula

$$\frac{p^2}{2m}$$

where p is the momentum, and m is the mass of the particle in question. If the particle is a proton or a neutron, then the uncertainty in kinetic energy is of the order of the nuclear binding energies. If, however, the particle would be an electron, the uncertainty in kinetic energy would be much bigger due to the small value of the mass of the electron. Actually, in this case the above formula would not even be applicable because the electron would move according to relativistic laws, but even calculating the energy by the relativistic formula $E_{kin} = cp$, it would be much higher than nuclear binding energies and the electron could not be held in the nucleus.

LA-24 (16)

November 11, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Second Series: Radioactivity Lecturer: E. Segre

LECTURE XVI: SURVEY OF STABLE NUCLEI

We shall now give a survey of the properties of the stable nuclei. The following table classifies the stable nuclei according to whether they contain an even or odd number of neutrons or protons.

N \ Z	even	odd
even	152	52
odd	55	4

The table shows that nuclei with an odd number of protons or an odd number of neutrons are relatively few in number. Nuclei containing both an odd number of neutrons and an odd number of protons are particularly few. There are only four such nuclei, all of them quite light. They are H^2, Li^6, B^{10}, N^{14}. It is seen that these nuclei form a simple series. It may be remarked that not only are there fewer kinds of nuclei with either an odd number of neutrons or an odd number of protons, but also that such nuclei are less abundant.

In connection with the fact that nuclei with an odd number of neutrons or an odd number of protons are relatively rare it may be mentioned that nuclei with an odd value of Z, i.e. with an odd number of protons, never have more than two isotopes. With the exception of the lightest elements, these isotopes differ by two neutrons. Similarly, if one considers nuclei with a given odd number of neutrons one will never find more than two stable nuclei of this kind and they will differ as a rule by two protons. On the other hand, nuclei with even Z may contain numerous isotopes.

Nuclei have been also classified according to whether their mass number A has the form $4n + 1$, $4n + 2$, or $4n + 3$. The class $A = 4n$ contains more nuclei than the others. The original reason for this classification was that the presence of α-particles with mass 4 was suspected in the nuclei. In a certain sense α-particles can actually be considered as sub-units in a nucleus.

Nuclei are frequently represented in diagrams as the following figure shows.

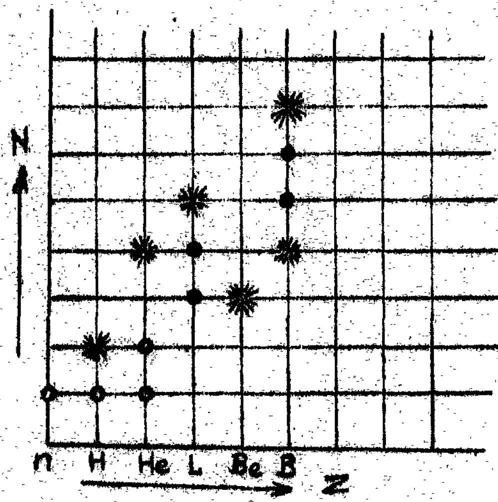


Fig. 1.

In this figure the abscissa Z is the number of protons and the ordinate N is the number of neutrons. Stable nuclei are indicated by dots, radioactive nuclei by crosses. The abscissa has been also provided with the appropriate atomic symbols.

If one draws the same figure on a smaller scale so that the whole periodic systems rather than its mere beginning may be represented one finds that the stable nuclei in this figure

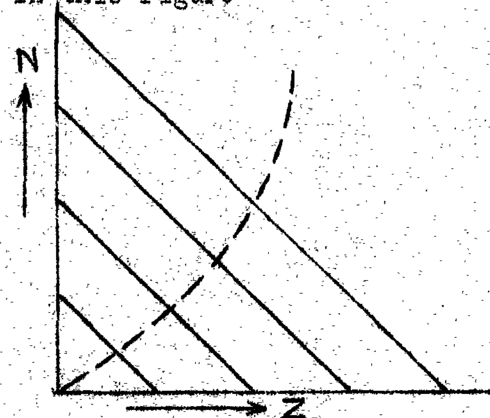
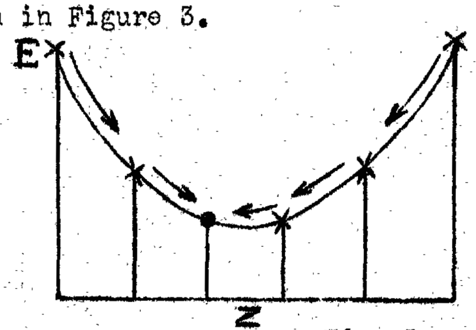


Fig. 2

would cluster around the thick dotted line shown in the figure. This line starts out at 45° corresponding to an equal number of neutrons and protons and then bends upward showing that heavy nuclei contain as a rule more neutrons than protons. The straight lines shown in the figure are lines connecting isobazic nuclei, that is, nuclei for which $A = N + Z$ has the same value. Such nuclei can in principle transform into each other by emission of electrons or positrons.

One might imagine the figure made more complete by adding the energy of the nucleus, that is, the energy needed to pull apart all the constituent neutrons and protons. One may plot this energy in the third dimension perpendicularly to the plane of Figure 2. The energy surface obtained in this manner will have a valley along the dotted line near which the stable nuclei cluster.

Cutting the energy surface along an isobaric line one obtains an energy curve as shown in Figure 3.



$Z + N = A =$ constant

Fig. 3

The straight lines shown in the figure are energies of the various isobaric nuclei. One of these energies will have the smallest value and a corresponding nucleus will be the only stable isobar. The corresponding stable nucleus is indicated by a dot while the unstable ones are shown as crosses. Arrows are also shown which indicate the direction in which nuclear β-transformations take place. The situation depicted in Figure 3 holds only for nuclei with an odd mass number. Isobars of such nuclei must either contain an even number of neutrons and an odd number of protons or else an even number of protons and an odd number of neutrons. There is no systematic deviation between the energies of these two kinds of nuclei. If the mass number is even, then the nuclei may either contain an even number of neutrons and an even number of protons, or else, an odd number of neutrons and an odd number of protons. The former kind of nuclei have systematically a lower energy than the latter kind. Thus the nuclear energies will lie on two roughly parallel curves, the higher one corresponding to the odd-odd case, the lower one to the even-even case. This is shown in Figure 4.

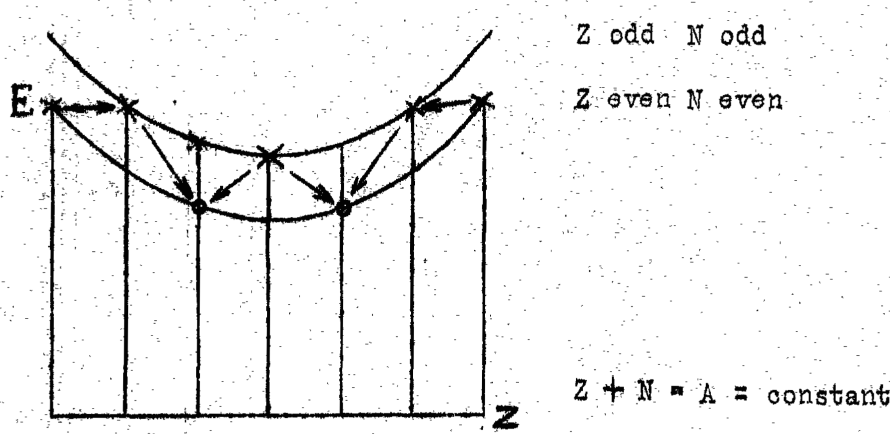


Fig. 4

It will be noticed that in this case more than one stable nucleus may exist. In the figure, two such nuclei are shown. The one on the right hand side has a higher energy than the one to the left. But the nucleus with the higher energy could transform into the one of the lower energy only by simultaneous emission of two charges which is an exceedingly improbable process. Emission of one charge (electron or positron) would carry us to the upper curve and would require energy rather than produce it. It will also be noticed from the figure that one nucleus on the higher curve may transform into a nucleus of lower energy either by shifting to the right or the left. The corresponding physical behavior is that such a nucleus emits positrons and electrons. This has actually been observed for several nuclei.

From the systematics of the nuclei, conclusions about nuclear forces may be drawn:

Equality of number of neutrons and protons.

This equality holds for the lighter nuclei. It is to be explained by assuming that the nuclear forces acting on neutrons and protons are similar. The explanation also uses the Pauli principle which asserts that in each state only one particle can exist: that is, in a given orbit within the nucleus one may have not more than two neutrons (or not more than two protons) differing in the direction of their spins. From the equality of the forces acting on neutrons and protons it follows that up to a given energy there will be a similar number of orbits available for neutrons and protons. If one then fills up the lowest orbits available up to a given energy, one will have to use a roughly equal number of neutrons and protons.

Greater number of neutrons in heavier nuclei.

This is to be explained with the help of the Coulomb law. It is easier to add a neutron than a proton to a nucleus which is already heavily charged. In lighter nuclei, the Coulomb forces are considerably smaller than the nuclear forces. In heavier nuclei the Coulomb energies of the several protons due to the longer range of these forces add up and become eventually comparable to the nuclear energies.

Mass defect and volume proportional to mass number.

These facts indicate forces having a saturation character, that is, a situation in which one particle contributes effectively to the total energy by interacting with a limited number of other particles. The same situation is encountered in crystals.

Binding energies of Deuteron and α-particle

The fact that the α-particle has a more than 10 times greater binding energy than the Deuteron shows that the saturation has not been reached at the latter particle. In the α-particle the saturation is actually reached. This shows that a particle, for instance a neutron, can effectively interact in a nucleus with another neutron and two other protons. The usual interpretation is that a particle can interact with all other particles which are permitted by the Pauli principle to move in the same orbit and that the interaction does not depend very strongly on the different spin directions which the particles may possess in this orbit.

The α Disintegration.

The α-particles emitted by a nucleus have a well defined energy and range. A very small fraction of the α-particle has sometimes a considerably longer range. The main fact of the uniform energy of the α's may be explained by assuming that the α-particle has been emitted by a nucleus of a given energy and that another nucleus of a given energy is left behind after the α-decay. The energy difference is then carried away by the α-particle.

Another important law of the α-radioactivity is the connection between the range of the particles and the decay constant. This is the Geiger-Nuttal relation. It states that the logarithm of the decay constant is a linear function of the range. Actually one finds that the relation between the logarithm of the decay constant and the range is slightly different for the radium, actinium, and thorium families. The three straight lines representing the relations for these three families are however parallel and their distances from each other are small. If one had plotted the logarithm of the decay constant as a function of the energy rather than as a function of the range, the plot would again be in good approximation a straight line.

The explanation of the Geiger-Nuttal relation is closely connected with the shape of the potential energy acting between the α-particle and the fraction of the nucleus which the α-particle has just left. At radii bigger than the nuclear radius r_0 shown in the figure, the interaction is a Coulomb repulsion as shown in the same figure. At distances closer than r_0 some kind of an attraction must exist, otherwise there would be no rea on why the original nucleus that has emitted the α-particle had stayed together any length of time. In the potential valley at distances smaller than r_0 the α-particle had in the original nucleus an energy level at the energy value E.

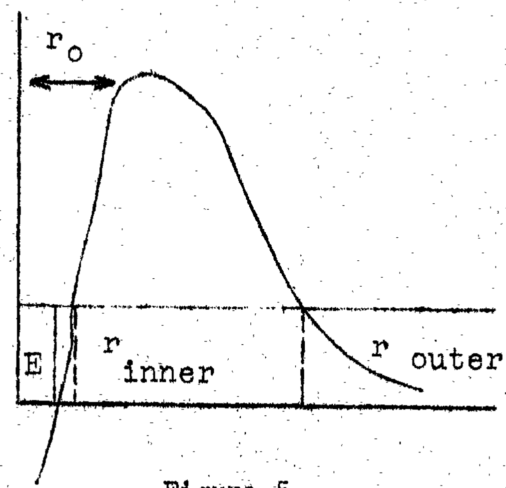


Figure 5.

After the α-particle has moved out to sufficiently great r values so that all interactions including the Coulomb interaction have become negligible, the α-particle will possess that energy E in the form of kinetic energy.

If now the product nucleus is bombarded by α-particles of energy E, pure Rutherford scattering is observed, indicating that at the radius r_{outer} to which the α-particle can penetrate according to classical mechanics, the unmodified Coulomb force still holds. There arises the question: in what way can the α-particle cross the potential barrier between r_{inner} and r_{outer} (see figure 5).

This question has been answered by Gurney and Condon, and by Gamow. In quantum mechanics leakage through a potential barrier is possible, though the probability of such a process becomes small when the barrier becomes high or broad. The time of decay may be estimated by imagining the α-particle oscillating within the potential hole, by calculating the number of times the α-particle bumps into the barrier per second, and finally, by multiplying with the probability that the α-particle penetrates the barrier in one collision.

The fact that the α-particle can penetrate through the barrier is connected with the α-particle's wave nature. An illustration from optics may be given. If light impinges on the interface between glass and air from the side of the glass under a sufficiently glancing angle, total reflection takes place, that is, all the light is reflected back into the glass and none penetrates into the air. Physical optics shows that actually a part of the wave field accompanying the light reaches out into the air. But it dies down exponentially, and at any appreciable distance it is not noticeable. Also it carries no energy stream. If now another glass plate is brought to within a few wave lengths of light from the first glass plate this second glass plate will be reached by the exponentially decaying field in air and part of the light will be transmitted to the second piece of glass.

The potential barrier, which α-particles cannot enter in classical mechanics, but which they can penetrate to a certain extent in wave mechanics, is analogous to the layer of air between the two glass plates. In both cases the amount of wave (light wave or α-particle wave) that can penetrate tends exponentially toward zero as the gap of the barrier becomes broader.

One might feel that it is absurd to expect an α-particle between r_{inner} and r_{outer} where the potential energy is greater than the total energy of the α-particle and that, on the other hand, the α-particle could not penetrate the barrier if it never were found between r_{inner} and r_{outer}. The answer is that if the appropriate experiment were made, the α-particle would be found sometimes on top of the barrier. But the energy needed for that would be given to the α-particle by the experimental procedure itself.

LA -24 (17)

November 16, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Second Series: Radioactivity Lecturer: E. Segre

LECTURE XVII: THEORY OF α-DISINTEGRATION. THEORY OF BETA PARTICLES

We have seen in the last lecture that classical physics could not explain the paradox of α-disintegration. The energy of α-particles from natural emitters is not sufficiently high to permit the α-particle to pass over the potential barrier surrounding the nucleus; hence the escaping particle must have passed through a region ($r_o < r < r_1$ of Fig. 1.) corresponding to negative kinetic energy. This phenomenon of barrier penetration has been explained by

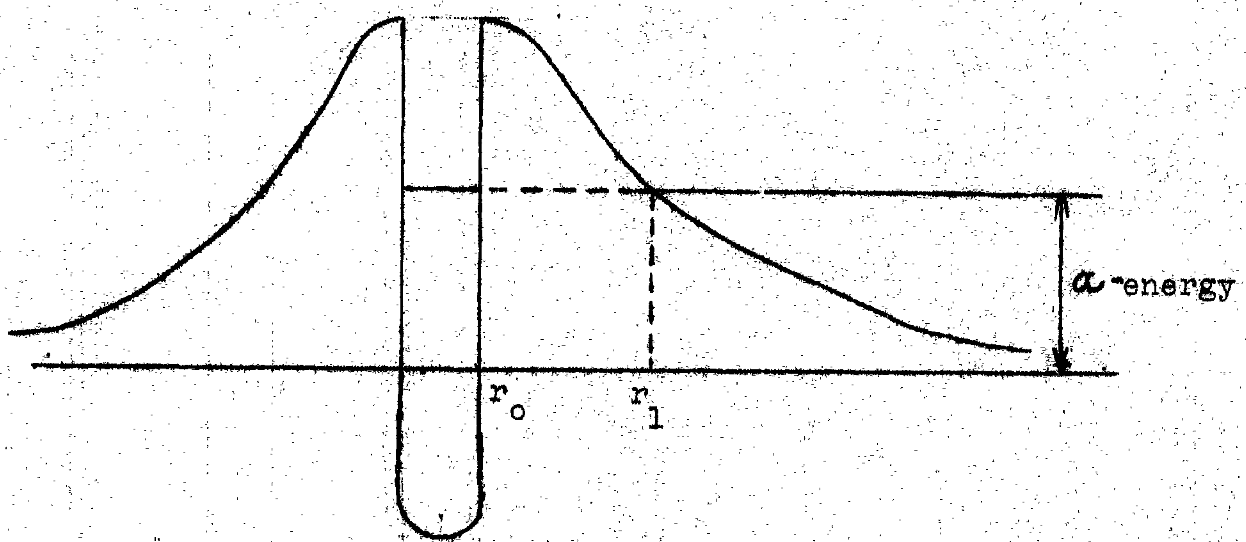


Fig. 1 Schematic potential of a nucleus.

quantum mechanics. It can be shown that there is a small but finite probability for a particle to transverse the "forbidden" region given by

$$P = e^{-\frac{2}{\hbar}\int_{r_1}^{r_o}\sqrt{2m_\alpha|E-U|}\,dr} \qquad (1)$$

where E is the energy of the emitted α-particle, U is the potential energy as shown schematically in Fig. 1; the integration is over the barrier width at the energy E, and h is Planck's constant. One can see qualitatively that the probability is smaller the greater the barrier height and the larger the barrier width, $r_1 - r_o$.

One may now obtain an expression for the disintegration constant based on the uncertainty principle. If r_o is the nuclear radius, and v the velocity of the α-particle, then approximately

$$\frac{h}{m\,v} \approx r_o$$

which may be written

$$v \approx \frac{h}{m_\alpha r_o}$$

The frequency with which the α-particle strikes the "potential wall" is then

$$\frac{v}{2r_o} \approx \frac{h}{2m_\alpha r_o^2}$$

Hence

$$\frac{h}{2m_\alpha r_o^2} e^{-\frac{z}{\hbar} \int_{r_o}^{r_1} \sqrt{2m_\alpha |E-U|} \, dr} = \lambda \qquad (2)$$

One can use the above formula to determine r_1 from the given total energy of the α-particle and the decay constant; a value

$$r_1 \sim 1.45 \times 10^{-13} \ A^{1/3}$$

is found, where A is the mass number of the nucleus. This value of r_1 is in agreement with that found from scattering experiments using the Rutherford formula.

An important application of Eq. (1) is in connection with artificial disintegration by charged particles, inasmuch as they must penetrate the potential barrier of the nucleus to initiate a nuclear reaction.

Anomalous α-tracks

On a Wilson cloud chamber photograph of an alpha source of some of the C products, one will occasionally observe a track considerably longer than the rest. This occurrence may be explained by a consideration of the so-called nuclear energy levels.

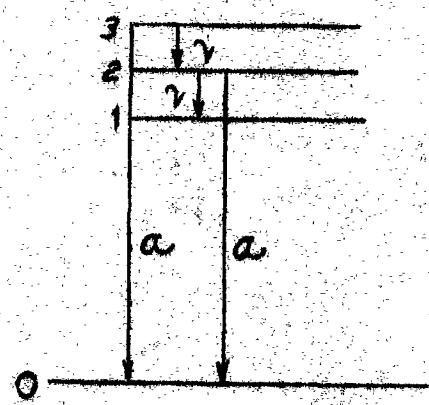


Fig. 2

Suppose initally all alphas are in the highest level, 3 of Fig. 2. A transition to the ground level may occur with the emission of a very energetic alpha. However, transitions to levels 2 and 1 may occur with emission of successive gamma rays and finally a transition to the ground state with the emission of a less energetic alpha. Since the γ-transitions are more probable, it is expected that the alphas will be of the less energetic type, and the more energetic (longer range) alphas will be much less frequent.

Theory of beta decay.

The most characteristic effect of β-decay is that the β-particles from a disintegrating nucleus have a continuous distribution in energy up to a well defined energy value. Experimentally a cloud chamber with a magnetic field reveals tracks of varying curvature corresponding to various energies of the β-particles. A typical spectrum is shown

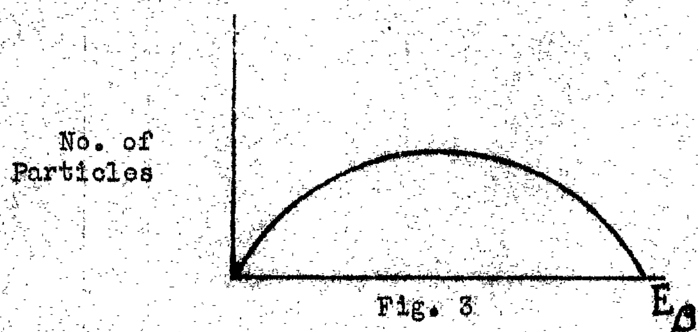


Fig. 3

in Fig. 3. This continuous distribution in energy is very surprising inasmuch as both the initial and final energy states are well defined. For instance, nuclei are known which emit in turn, an alpha, a beta, and then another alpha. The two alphas have quite definite energies, yet the beta particle does not. Some years ago, attempts were made to measure the energy of the betas by using a very thick Pb walled calorimeter. It was hoped at the time that all the energy of the emitted particles would be measured. However, an amount, corresponding only to the area under the distribution curve, was found.

Bohr made the suggestion that perhaps energy and momentum was not conserved in β-disintegration. A special hypothesis made by Pauli was that a second particle -- the neutrine -- was emitted, so that conservation laws were still valid.

Attempts have been made to correlate the maximum energy E in a beta spectrum with the disintegration constant λ, as was done in the case of α-particles (Geiger-Nutall laws). The Sargent curves, a plot of log λ vs E, are an attempt in this direction.

From absorption measurements in aluminum and using a standardized method of extrapolating the range-energy curve, one obtains

$$R = 0.543\,E - 0.16$$

where E is in million electron volts and R in cm. (Feather relation).

A satisfactory theory of beta-disintegration must explain the continuous spectrum and give a correct range-energy relation. Fermi has developed a theory with the neutrino hypothesis. In the nucleus a neutron is converted to a proton together with the emission of an electron. The role of the electron is similar to that of a photon in radiation theory. The probability of emission of an electron with energy in the interval E, E + ΔE for light nuclei is given by

$$P(E)dE = g^2 \frac{m_e^5 C^4}{2\pi^3 \hbar^7} \left| \int u_n^x v_m d\tau \right|^2 (E_o - E)^2 \sqrt{E^2 - 1}\ EdE$$

where E is in units of $m_o c^2$, C the velocity of light, E_o the maximum β-particle energy, $\hbar$ is Planck's constant divided by 2π, g is a universal constant of the theory. Attempts have been made to correlate it with other known constants (e.g., mass of mesotron). u and v are nuclear eigenfunctions; the integral is the so-called matrix element and corresponds to expressions in radiation theory related to the transition probabilities. For heavy nuclei, one must take into account the large Coulomb interaction of the beta-particle with the nucleus. The shape of the distribution curve at the high energy end depends on the mass of the neutrino. Experimental evidence indicates a relatively very small mass for it. Several features of the detailed theory can be checked on the β-decay of nuclei containing z protons and z - 1 neutrons such as C^{11} and N^{13}, λ calculated from the theory agrees quite well with experiment. (White, Phys. Rev., 59, 63 (1941).

The neutron is expected to be β-active with a mean life of several hours. Experimental evidence is lacking because the neutron reacts relatively quickly with any nucleus that may be present. However, experiments have been proposed to measure this disintegration constant.

LA-24 (18)

November 18, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Second Series: Radioactivity Lecturer: E. Segre

LECTURE XVIII: 1) POSITRON EMISSION AND K CAPTURE 2) GAMMA RADIATION

1) Positron Emission and K Capture

Nuclei are known which emit positrons, giving rise to isobars with one less nuclear charge than the parent nuclei. This reaction occurs if the available energy is at least 500 kv, the energy equivalent of the positron, that is, if the mass of the parent nucleus exceeds that of the daughter by at least the mass of the positron. If the energy available is less, then the nucleus will capture an electron from the innermost shell and subsequently emit an X-ray quantum. This so-called K (electron)-capture by the nucleus also occurs at the higher energies but is less possible.

2) Gamma Radiation

The electromagnetic radiations emitted by radioactive substances are called γ-rays. The γ-rays are characterized by their frequency, or as is more usual by the energy of a quantum expressed in electron volts or in units of $m_e c^2$. The range in energy of γ-rays from naturally radioactive substances is approximately 50,000 eV to about 3 Mev (million electron volts). By bombarding Li with protons more energetic γ-rays are obtained, values up to 17 Mev have been found. The highest energies in the lab have been reached with the Kerst betatron.

The theory of γ-ray emission is similar to that of the emission of light from atoms and molecules, except that the magnitudes of some of the quantities involved are much different. The probability of a transition from one energy state say m, to another lower energy state n, and the emission of a light quantum is given by

$$W = \frac{\omega^3}{3C^3} \quad P^2 \qquad (1)$$

where ω is 2π times the frequency of the emitted radiation, C the velocity of light. P is the so-called matrix element of the transition and is the electric moment averaged over the eigenfunctions of the two states m and n. Equation (1) represents the first term of an expansion in powers of r_o/λ where r_o is the nuclear radius and λ the wavelength of the emitted quantum. Physically, the first term in the expansion corresponds to the dipole radiation, the second to the quadrupole radiation, etc. In atomic processes, λ is so large compared to the radius of the atom, so that the second term is very much smaller than the first. For nuclear processes, however, λ and r_o do not differ as greatly; consequently the second term makes an appreciable contribution. The magnitudes of the matrix elements vary over a considerably wide range corresponding to wide variations in the intensity of spectral lines. It sometimes happens, for reasons of symmetry, that the matrix element for dipole radiation is exactly zero, that is, the dipole transition is forbidden. In this case, the transition probability is governed by the second, or quadrupole term. For atomic processes we have seen that quadrupole transitions of appreciable transitions are rare, but for nuclei, are quite frequent.

The dipole transitions correspond to changes of one unit of angular momentum, and for quadrupole transitions to a change of two units.

In some nuclear transitions the nucleus will interact with the electrons surrounding the nucleus - with the result that an electron is ejected whose kinetic energy is given by the difference of the excited nuclear energy level and the energy of excitation of the particular electron. If a K electron is ejected, then

$$E_{kin} = E_{nucleus} - E_K$$

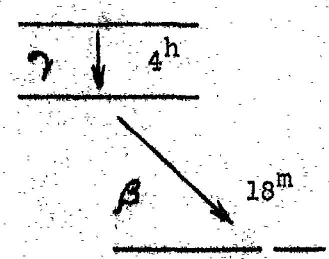
This process is known as internal conversion. Photographs on β-spectrograph show that the internal conversion electrons, as is to be expected, form very homogeneous groups in contradistinction to disintegration electrons which have a continuous distribution in energy. Usually a group of lines of varying intensity is observed corresponding to ejection of electrons from different energy levels of the atom.

Occasionally γ-radiation is accompanied by β-disintegration and the question arises as to whether the radiation precedes or follows the disintegration. If internally converted electrons are present and are analyzed to determine the energy differences between the various groups, an answer can be given. For if the spacing corresponds to an X-ray diagram of the parent nucleus, then the γ-radiation precedes the disintegration, whereas if the spacing is that of the daughter nucleus, the converse is true.

The ratio of the number of conversion electrons to the number of γ-quantum is known as the conversion coefficient.

Nuclear isomerism

If the transition probability from the first excited nuclear level to ground level is relatively very small so that the mean life of that level is appreciably large, the excited level is said to be metastable. Let us consider some experimental evidence. If the nuclei of Br^{80} are in the ground state, they undergo β-disintegration with a half-life of 18^m. If on the other hand, the nuclei are in the first excited level, which has a half-life for γ-emission



of 4^h, then the observed half-life of the subsequent β-emission is found to be also 4^h, since the rate determining factor is the slower rate of γ-emission. A further proof of such "isomeric" nuclei is that they can be produced by excitation by γ-rays or electrons, first to a higher level, from which they reach the metastable level by γ-emission of smaller energy.

Surprisingly enough, rather simple chemical techniques are known for separating the so-called nuclear isomers. About fifty such nuclei are known.

LA-24 (19)

November 30, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Third Series: Neutron Physics   Lecturer: J. H. Williams

LECTURE XIX: DISCOVERY OF THE NEUTRON

The first evidence of the existance of the neutron was found by Bethe and Becker (Zeits f. Physik 66, 289 (1930), (Naturwiss 19, 753 (1931)). They published the results of some experiments in which the light elements Li, Be, and B were bombarded by polonium alpha particles. A very penetrating "radiation" was emitted which was more difficult to absorb by lead than the shortest wave length γ-rays then known, - those of ThC". Their original explanation was that the radiation was light of a very short wave length. Today we know that the increased absorption of very short wave-length γ-rays by the process of pair creation would eliminate this explanation. They were lead to the assumption that the "radiation" was photons on other grounds, besides that of absorption measurements, one being that no tracks were observed in a cloud chamber.

Carrying their hypothesis of hard γ's to its logical conclusion on the basis of the Klein-Nishina absorption curve as a $f(E_\gamma)$, these early experimenters gave an energy for this "radiation" of from 7 to 15 Mev. Their absorption measurements were made with Geiger counter detectors. In a nuclear equation they postulated that the reaction involved was

$$_4Be^9 + {}_2He^4 \longrightarrow {}_6C^{13} + h\nu$$

Suppose for an exercise we check the energy balance of this equation by substituting masses in the equation

$$\begin{aligned} {}_4Be^9 &= 9.01504 \\ {}_2He^4 &= \underline{4.00389} \\ &\ 13.01893 \\ C^{13} &= \underline{13.00761} \\ &\ \ \ .01132 \text{ mass units} \end{aligned}$$

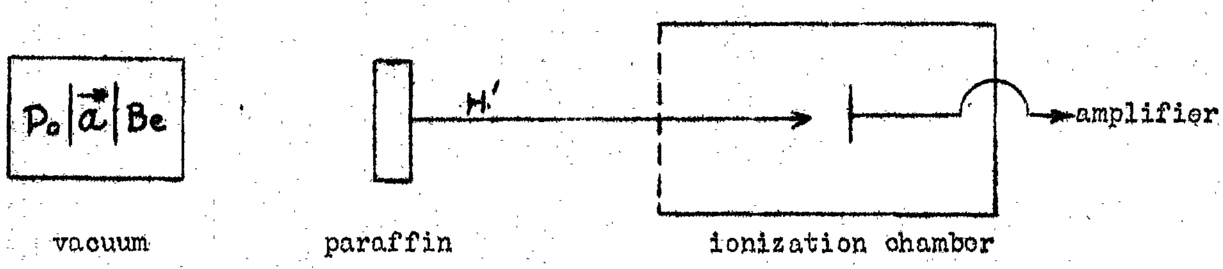
One mass unit = 931 Mev

.01132 " " = 10.5 Mev

Energy available = E_α + 10.5 Mev.

The problem was next attached by Curie and Joliot (Comptes Rendus 194, 273, 708 (1932)). The important difference was that in these researches, thin window ionization chambers instead of Geiger counters were used as detectors of the radiation. They found that if material containing H were interposed between the α on Be source and the ionization chamber, a large increase in ionization was observed. For absorbers of other kinds only small decreases were observed. They showed that this increase in ionization was due to the presence of H recoils leaving the "absorber" and entering their ionization chamber. Cloud chamber studies revealed the presence of these proton recoils as well as He recoils in a He gas. By measuring the lengths of these recoil tracks it was possible to determine the energy of the recoils. Some of these recoil protons had energies up to 5 Mev. No tracks were observed for the primary "radiation" which they assumed to be γ-rays. Curie and Joliet explained these observations by saying that the energy transfer occurred in a Compton process. The γ-ray energy calculated in this way turned out to be 50 Mev. Unfortunately, the calculations for other recoil particles on the basis of this theory gave different γ-ray energies. They were higher for heavier recoil nuclei.

Chadwick (Proc. Roy. Soc. A. 136, 692 (1932),) solved this dilemma so convincingly that he received the Nobel prize. A schematic diagram of the apparatus used follows.



The measured recoil energies of H' and N^{14} nuclei were 5.7 Mev and 1.2 Mev respectively. On the basis of the photon hypothesis the calculated "radiation" energy would be 55 Mev and 90 Mev respectively.

Thus the photon hypothesis was inconsistent with the observations. Either the laws of conservation of energy and momentum or the hypothesis as to the nature of the phenomenon was to be rejected. Chadwick therefore assumed that the "radiation" was a new particle of mass $\simeq$ mass of proton and having zero charge. He christened it the neutron.

Consider a head-on collision between equal mass particles. Then from the conservation of energy we have

$$1/2\ MV_1^{\,2} = 1/2\ MV_2^{\,2} + 1/2\ MV_r^{\,2} \qquad (1)$$

and from the conservation of momentum we have

$$MV_1 = MV_2 + MV_r \qquad (2)$$

Solving $V_2 = V_1 - V_r$

Substitute in (1)

$$V_1^{\,2} = V_1^{\,2} - 2V_1, V_r + V_r^{\,2} + V_r^{\,2}$$

$$0 = -V_1V_r + V_r^{\,2}$$

$$V_r = V_1$$

$$V_2 = 0$$

All the energy of the neutron is therefore passed on to the proton in a head-on collision. Thus, if the maximum recoil energy of the protong was 5.7 Mev the neutron energy was 5.7 Mev.

If we consider the case of the recoiling N^{14} nucleus by the same equations as above with the condition that $M = 1/14$ we find

$$V_r = \frac{2M}{M_r + M}\ V_1$$

and $E_r \simeq 1/4\ E_n$

This is in fair agreement with Chadwick's result and further measurements served to substantiate his hypothesis as to the character of the neutron.

Zero charge means practically zero ionizing power - one ion pair 3 meters path in air. Thus we explain the observation that the neutron leaves no visible track in a cloud chamber.

The neutron loses energy principally by interaction with nuclei and not appreciably to the electrons of the matter through which it passes. These mutual forces between nuclei are the subject of Mr. Critchfield's lectures.

Artificial Sources of Neutrons

$$_4Be^9 + {}_2He^4 \rightarrow {}_6C^{13} \rightarrow {}_6C^{12} + {}_0n^1 + Q$$

Not a source of monoenergetic neutrons.

This mode of disintegration with α-particles as the bombarding particle which results in neutron production has been observed for several light elements, Li^7, B^{11}, N^{14}, F^{19}, Na^{23}, Mg^{24}, Al^{27}, but is most intense from Be, then B and Li.

These sources of neutrons are essentially very weak, relative to sources of which we shall speak later. It was impossible up to 1942 to get sources in this manner using Po as the α-source which give more than $\sim 10^5$ r/sec or expressed in another way the yield is 50 n/10^6 α's stopped. On the other hand, if the α's are obtained from R_a which is intimately mixed with Be, the number of neutrons emitted per sec. per mc. of Rn is of the order of 10^4/sec. With a curie of Ra one therefore has an appreciably active source. Unfortunately γ-rays are present here where they are absent in the Po α sources, except $\sim 1\ h\nu/n$.

A complete discussion of α-n reactions, the energies of the neutrons and of the associated γ-rays is given in Livingston and Bethe, Rev. Mod. Phys. 9, pages 303 to 308, 1937.

Photoelectric Production of Neutrons.

With two of the light elements $H_1{}^2$ and $Be_4{}^9$, neutrons can be produced by the γ-rays of ThC" or any γ-rays whose energy exceeds the binding energy of the neutron in the primary nucleus. The probability of this process is low and as a consequence the yield of these sources is correspondingly low.

Examples - $H^2 + h\nu \rightarrow H^1 + n^1 + Q$

Chadwick and Goldhaber, (Proc. Roy. Soc. 151, 479 (1935),) used the ThC" γ-rays of energy 2.62 Mev and measured the actual magnitude of the ionization of the protons formed in the above reaction in order to determine their energy.

Since $E_n = E_p$ as consequence of their having approximately equal masses, the energy equation means

$$E_\gamma - 2E_p = -Q \quad 2.62 - .45 = \underline{\underline{2.17 \text{ Mev}}}$$

The analysis of the data gave Q =-2.2 Mev

This means the neutron is bound in the H^2 nucleus with an energy of 2.2 Mev.

Knowledge of this binding energy is of great theoretical importance since it allows us to determine the mass of the neutron.

<u>Mass of Neutron</u>

Rewriting the photo-disintegration equation

$${}_1H^2 + hv \longrightarrow {}_1H^1 + {}_0n^1 \quad \text{as a mass equation}$$

$$2.01473 + \frac{2.62}{931} = 1.00813 + M_n + \frac{.45}{931}$$

$$M_n = 1.00893 \pm .00005$$

A table in Livingston and Bethe, page 353, summarizes the evidence on γ-α reactions up to 1937.

A modern useful source of neutrons using γ-rays on light elements illustrates the advances in technique made in nuclear physics in recent years.

As we will discuss later, it is possible to produce an isotope of Y by deuteron bombardment of Sr which has a 1/2 life of 80 days and emits a γ-ray of sufficient energy to dissociate Be^9 and give neutrons of 160-kev energy. This source has been extremely useful as a source of relatively monoenergetic neutrons.

LA-24 (20)

December 2, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Third Series: Neutron Physics   Lecturer: J. H. Williams

LECTURE XX   I. Neutron Sources (continued)

II. Typical Reactions for Production of Neutrons

Neutron Sources (Continued). Neutrons may be produced by the bombardment of certain nuclei with protons, deuterons or alpha particles, that is, a nuclear reaction takes place and a neutron is emitted. To initiate the reaction, the bombarding particle must penetrate the potential barrier surrounding the target nucleus. In other words, the projectiles must have a sufficiently high kinetic energy.

We may distinguish two methods used to accelerate such particles, namely direct and resonance accelerators. The first type is simply a long vacuum tube, one end of which is at a very high potential and contains a source of ions, and the other is usually at ground potential and contains the target. The ions traverse the vacuum tube acquiring kinetic energy corresponding to the drop in potential.

The resonance accelerator, or cyclotron, consists of a large, flat, hollow cylinder separated into two sections, each resembling the letter D. An oscillating voltage difference is applied between the two "dees" of the order of a hundred kilovolts and at a frequency of several magacycles. Along the axis, a magnetic field of several thousand gauss is applied. The ion source is on the axis. Under the proper conditions, the ions travel in circular orbits of increasing radii and increase their kinetic energy during each passage from one of the dees to the other. The success of the cyclotron is due to the fact that the angular frequency of the motion is independent of the energy (i.e., of the radius of curvature of the orbit). When the voltage frequency is equal to the angular frequency, resonance conditions obtain. It is possible by such resonance means to attain energies of the order of several million electron volts.

<u>Typical Reactions for Production of Neutrons</u> <u>(p,n) reaction:</u> In this type, the incoming particle is a proton and the emitted one a neutron. Written in detail:

$$Z^A + H' \longrightarrow (Z + 1)^A + n' + Q$$

where A is the mass number and Z the atomic number. Q is the energy of the reaction. It turns out that Q in this case is negative, that is, that energy must be supplied to the reaction. In the first place, the neutron differs in mass from the proton by an energy equivalence of 0.8 MEV, so at least that much kinetic energy of the proton would be necessary (assuming the ground levels of the two nuclei, Z^A, $(Z + 1)^A$, are equal). If it is known that the $(Z + 1)^A$ is a beta-emitter, then an additional million electron volts (equivalent to two electronic masses) are required. For the case

$$Li^7 + H' \longrightarrow Be^7 + n'$$

the threshold energy is 1.86 MEV. The energy of the emitted neutron in the forward direction in the center of gravity system of coordinates will be roughly the difference of bombarding energy and 1.86 MEV. Thus this method is very convenient for producing mono-energetic neutrons simply by controlling the bombarding voltage. It is clear from consideration of momentum and energy conservation that for a given bombarding voltage, the energy of the emitted neutron depends on the direction of emission, so that neutrons observed in the backward direction of the beam will have less energy than those in the forward direction, so that one may obtain a range of neutron energies as a function of angle. Practically, there is an effective spread of bombarding energies because of the fact that some of the neutrons will be slowed down in the finite thickness of the target before they initiate a nuclear reaction. This variation can be made arbitrarily small but is usually of the order of 5 to 50 kv for intensity reasons.

<u>(d,n) reaction:</u> Here the type is

$$Z^A + H^2 \longrightarrow (Z + 1)^{A+1} + n^1 + Q$$

The so-called (d,d) reaction is an example:

$$H^2 + H^2 \longrightarrow He^3 + n' + 3.2 \text{ MEV}$$

This reaction provides a copious source of neutrons. Yields of the order of 10^8 neutrons/sec are possible with bombarding currents of the order of 10 μA impinging on heavy ice targets. The efficiencies are of the order of $1n/10^5$ deuterons. A desirable feature is that the reaction has an appreciable cross-section at low bombarding energies. One difficulty is the preparation of thin targets. Often gas targets are used but the yields are correspondingly smaller. Another example is

$$C_6^{12} + H_1^2 \longrightarrow N^{13} + n' + Q$$

with Q = -.25 MEV, which means that the reaction is endoergic. The energy spectrum of the emitted neutrons is not simple because of the $C^{13}(d,n)N^{14}$ reaction.

Disintegrations produced by neutrons (n,a) reactions: An example of this type is

$$n' + N^{14} \longrightarrow He^4 + B^{11}$$

One can obtain cloud chamber photographs of this process to establish its identity beyond question, but it is a somewhat tedious procedure since many photographs must be taken before one finds evidence for the reaction. Another well known reaction is

$$n' + B^{10} \longrightarrow He^4 + Li^7$$

and also

$$n' + Li^6 \longrightarrow H^3 + He^4$$

Both of these reactions have relatively very large cross-sections for thermal neutrons and are very useful for the detection of neutrons by detecting the ionization produced by the emitted α-particles. A common procedure is to line the walls of the ionization chamber with a layer of boron. Often proportional counters are used which contain gasious boron trifluoride and the individual particle pulses are detected.

(n,p) reaction: Symbolically this reaction is

$$Z^A + n' \longrightarrow (Z - 1)^A + H^1$$

This reaction is usually endoergic, the case $Z^A = N^{14}$ being an exception.

(n,γ) reaction: This type is the simple capture process in which a neutron is absorbed and a gamma quantum omitted. Process was first observed by Fermi in the production of artificially radioactive isotopes by slow neutron irradiation.

<u>(n, 2n) reaction:</u>

$$Z^A + n' \longrightarrow Z^{A-1} + 2n$$

It is clear that this reaction requires at least an amount of energy corresponding to the binding energy of a neutron in the nucleus. Hence fast neutrons are required to make reaction go. This process can only be reliably identified if the resulting nucleus is radioactive.

LA-24 (21)

December 7, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Third Series: Neutron Physics   Lecturer: J. H. Williams

Lecture XXI Properties of Slow Neutrons

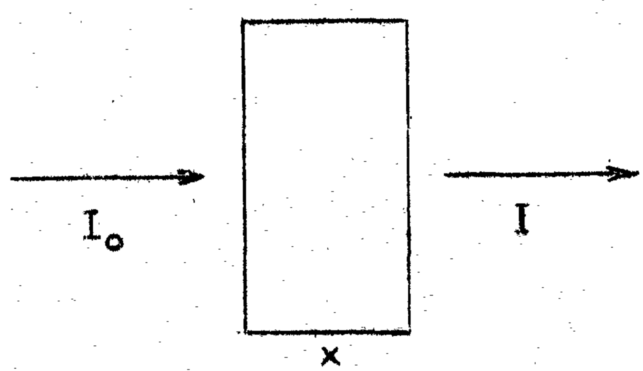
Slow neutrons are produced by allowing fast neutrons to pass through paraffin. The collisions of the neutrons with the protons in paraffin result in the initial energy of the neutron being divided up between the neutron and the proton. Thus in successive collisions the neutron loses energy until it finally winds up with thermal energies.

The process of slowing down of the neutrons depends on the collision cross-section between neutrons and protons. We shall discuss the concept of a collision cross-section, as it has been introduced in the kinetic theory of gases.

Let us assume that two gas atoms collide if they come closer to each other than the distance s. Assume further that the relative velocity of the two particles is V. Then, the particle under consideration will sweep out per unit time a volume $\pi s^2 V$ and the particle will collide with all collision particles within this volume. Therefore the number of collisions per second is $\pi s^2 V n$ where n is the number of collision partners per unit volume. Assuming that all of the relative velocity V is due to the motion of the incident particle, the distance through which this particle travels between collisions, that is the mean free path, is given by

$\ell = 1/\pi s^2 n$. (in the kinetic energy of gases the incident particle and its collision partners move with the same average velocity which introduces into the formula of the mean free path a factor $1/\sqrt{2}$.)

The probability that a particle should move through a distance x without making a collision is given by $e^{-x/\ell} = e^{-x\pi s^2 n} = e^{-x o n}$. Here the collision cross-section $o = \pi s^2$ has been introduced. This cross-section is in gas kinetic theory the geometric area within which collisions take place. Experimentally, the above exponential formula may be used to measure the cross-section. For this purpose a slab of the material made out of the collision particles and having thickness x is placed into a beam of incident neutrons with intensity I_o. A smaller intensity I will emerge from the slab since the neutrons which made the collision are missing from the beam. Then the ratio of the two intensities is given by the formula $I/I_o = e^{-nox}$

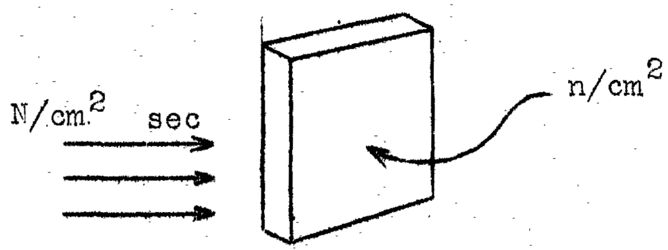


Nuclear cross-sections have usually the order of magnitude $10^{-24} cm^2$. This quantity has been introduced as a unit of nuclear cross-sections. It is called the barn.

The collision cross-section and mean free path of neutrons in paraffin vary a great deal with the energy of the neutrons. The cross-section is a few barns above a Mv whereas it is fifty barns for thermal energies. At the high energies the mean free path is a few cm, while thermal neutrons have a mean free path of a few mm.

The energy loss of a neutron in a collision with a proton depends on the angle of deflection. If the initial energy of the neutrons is E_o then the average energy of the neutrons after n collisions is E_o/e^n. Thus approximately 20 collisions are necessary to reduce fast neutrons to thermal energy.

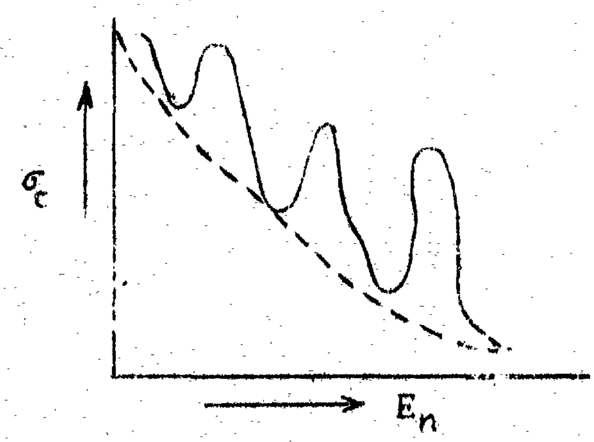
The number of slow neutrons that can be obtained in a given volume of paraffin depends on the rate of production of slow neutrons and on the rate of absorption of slow neutrons by the hydrogen in the paraffin. This absorption is due to the process $H^1 + n^1 \longrightarrow H^2 + h\nu$. This reaction is only one example of the very common process in which a slow neutron is absorbed by a nucleus and a γ-ray is emitted. The resulting nucleus is frequently radioactive. (This is not the case when neutrons are absorbed by H^1.) The production of radioactive nuclei provides us with an easy means of measuring the capture cross-section, i.e., the part of the collision cross-section which is due to the capture process.



To perform the measurement we use a thin sheet containing n absorbing atoms per cm^2. We let N nuclei impinge upon the sheet per cm^2 and second. Then the number of radio-active nuclei formed per cm^2 and second is given by nNo_o. Thus the capture cross-section may be measured by determining the radio-activity formed.

Some nuclei have very high thermal capture cross-sections. For instance, the capture cross-section of cadmium is 3,000 barns. While this cross-section is very much greater than the geometric cross-section of the nucleus, the scattering cross-section at corresponding energies is only a few barns, that is, comparable to the geometric cross-section.

A further peculiarity of the capture cross-sections at low energies was found by Moon and Tillman (Proc. Roy. Soc. A 153 467 (1936)). They observed that the capture cross-section changes rapidly with the neutron energy showing maxima as can be seen in the following figure.



The maxima are separated by distances of the order of 10 electron volts on the energy scale.

The crude explanation of this phenomenon is resonance. The word resonance suggests the analogy with the great energy transfers which become possible between vibrating systems if the frequencies are equal to each other. If the rest energy of the initial nucleus, plus the rest energy of the neutron, plus the kinetic energy carried in by the neutron, happens to agree with the energy of a compound nucleus made up from the initial nucleus and the neutron, then the reaction becomes more probable. For instance, the above reaction of neutron capture by hydrogen may be written in the form $H^1 + n^1 \longrightarrow H^2_* \longrightarrow H^2 + h\nu$ where the compound nucleus H^2_* is an excited state of the nucleus H^2. Not only capture but also scattering may be described with the help of compound nuclei. For instance, the scattering of neutrons by protons may be written in the following form $H^1 + n^1 \longrightarrow H^2_* \longrightarrow H^1 + n^1$. The various means by which the compound nucleus may disappear, that is, emission of a quantum or emission of a neutron, or possibly emission of another particle, compete with each other. At low neutron energies emission of a γ-quantum and consequent neutron capture predominates for most nuclei.

The first theory of resonance capture of neutrons was given by Breit and Wigner (Phys. Rev. <u>49</u> 519 (1936)). Rather than discuss the complete theory we shall describe the behavior of the neutrons

A) if their energy is very close to a resonance

B) if their energy is very low.

In this way, we shall understand the general appearance of the curve for σ_c which has been given above, that is, the general rise of σ_c with decreasing energy (B) and the superposed resonances (A).

If condition (A) is fulfilled, the collision cross-section is determined by the single neighboring resonance with energy E_r. Then, according to Breit and Wigner a neutron with energy E possesses a collision cross-section

$$4\pi \lambda\!\!\!^{-} \lambda\!\!\!^{-}_r \frac{\Gamma_n \Gamma}{(E-E_r)^2 + \Gamma^2}$$

Here $\lambda\!\!\!^{-}$ is the de Broglie wave length divided by 2π of the neutron of energy E and $\lambda\!\!\!^{-}_r$ is the same quantity for the case that the neutron possesses the resonance energy E_r. Γ is the width of the resonance and Γ_n is that part of the width which is due to the neutron emission. A detailed explanation of these quantities will be given in a later lecture.

If $E - E_r$ is great compared to Γ, the collision cross-section will fall off inversely as the square of $E-E_r$ at both sides of the resonance. In exact resonance only the Γ^2 term remains in the denominator and at this point (or rather close to this point) the cross-section attains its maximum.

If on the other hand, the energy of the neutron is small compared to all resonance energies (assumption B), then the energy dependence of the cross-section is determined by the variation of $\lambda\!\!\!^{-}$ with energy. In this region, the cross-section varies as 1 over the velocity of the neutron. The size of this region depends on the spacing of the resonance levels. When the first resonance level is reached the 1/v law breaks down.

LA-24 (22)

December 9, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Third Series: Neutron Physics Lecturer: J.H. Williams

LECTURE XXII: PROPERTIES OF SLOW NEUTRONS (continued)

If the width of the lowest level Γ , is greater than the energy E_r of the lowest level, then the 1/v law will break down when the neutron energy becomes comparable to the width Γ .

The above statements concerning the region of validity of the 1/v law can be proved from the energy dependence of the capture cross-section as given by the Breit-Wigner formula (see end of last lecture). This energy dependence has the form

$$\sigma_o \sim E^{-1/2} \Big/ \left\{(E - E_r)^2 + \Gamma^2\right\}$$

One sees that σ_o will behave as 1/v if the condition

$$d/dE \log E^{-1/2} \gg d/dE \left\{(E - E_r)^2 + \Gamma^2\right\}$$

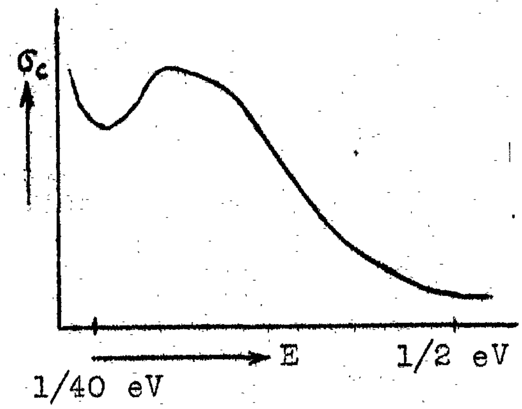
is satisfied. This in turn is proved if either

$$E \ll E_r$$

or

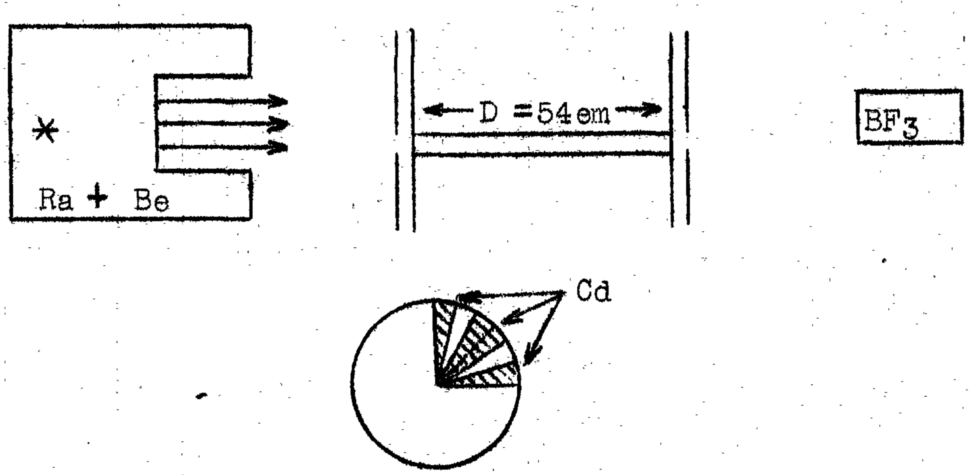
$$E \ll \Gamma$$

One very important substance in slow neutron physics is cadmium. Its absorption cross-section as a function of energy is roughly represented by Fig. 1.



Its strong absorption for thermal neutrons is due to a resonance lying very close to thermal energies. A few tenths of a millimeter of Cd shielding is sufficient to remove from a distribution of neutrons the thermal neutrons.

The first experiments, by which information on the Cd absorption and on the distribution of thermal neutrons was obtained, were performed at Columbia. The experimental arrangement is shown in Fig. 2.

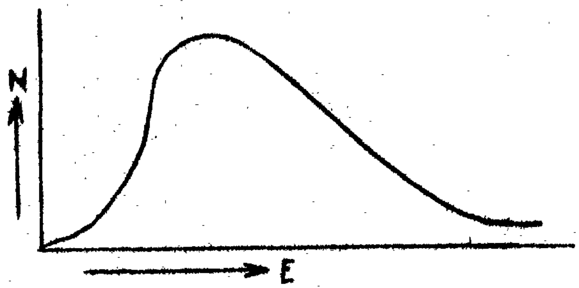


On the left side of this figure a paraffin "howitzer" is shown, which is a block of paraffin shaped as shown in the figure and containing a radium-beryllium source at the indicated position. The paraffin block emits slow neutrons from all of its surfaces but the arrows indicate a particularly strong and somewhat better directed beam of slow neutrons produced by the peculiar shape of the howitzer. This neutron beam impinges upon a velocity selector which is shown in the figure to the right of the howitzer. This velocity selector consists of two rotating discs mounted on an axis at a distance D from each other and two stationary discs located before the first and behind the second rotating disc. On each of the four discs, Cd segments are mounted in such a way that there are on every disc 50 Cd segments and 50 segments free of Cd. This arrangement is shown schematically below the velocity selector in the figure. To the far right of the figure a boron trifluoride counter is shown which detects the slow neutrons that have passed through the velocity selector.

The velocity selector is rotated with a velocity of n revolutions per second. The Cd strips of the rotating and the adjacent stationary discs will at a given time overlap, so that the neutrons can get through the Cd-free parts of the disc. Shortly thereafter, the Cd segments on the rotating and stationary discs will alternate so that no neutrons can get through. If the velocity of the neutrons is such that they will find either the first or the second pair of rotating and stationary discs closed, then a minimum of transmitted neutron intensity will be observed. Assuming that all neutrons have the same velocity v, this minimum should occur when the relation

$$v = 100n/D$$

is satisfied. It has been found that the neutrons emitted by the howitzer have an approximate velocity of 2,500 meters per second. This corresponds to thermal velocities. A somewhat more detailed analysis showed that the neutron velocities have approximately a Maxwellian distribution shown in the figure.

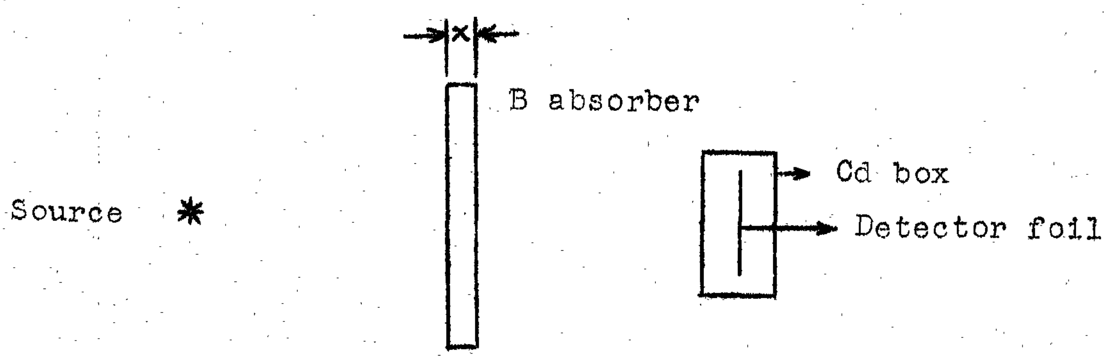


In this figure, the number of neutrons having a given energy E is plotted against that energy. The maximum of that curve shifts to higher energies when the temperature is raised.

An investigation of the resonances occurring at a few eV was first carried out by a somewhat indirect method. This method takes advantage of the fact that the reaction

$$n^1 + B^{10} \longrightarrow He_2{}^4 + Be_4{}^7$$

has a cross section obeying the 1/v law up to quite high energies. The experimental arrangement is shown in Fig. 4.



On the left hand side is indicated a neutron source (this might be a howitzer or another arrangement). The neutrons emitted by this source pass through a boron absorber of thickness x. Experiments are carried out with various thicknesses x. On the right hand side of the figure is seen a detector foil which is made of a substance that becomes radioactive under neutron bombardment. It is in this substance that we want to study the neutron capture resonances. The detector foil is enclosed in a Cd box. Four kinds of measurement are made:

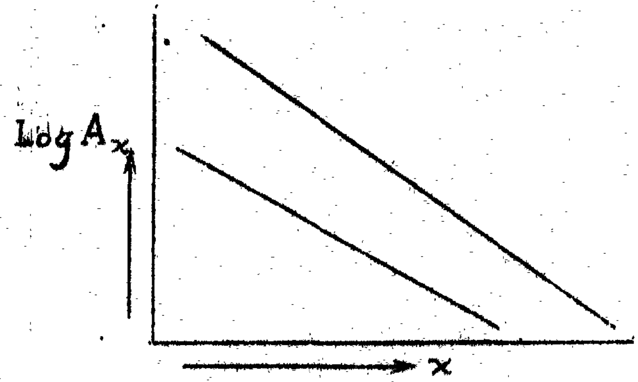
1. no boron absorber present, no cadmium present.
2. no boron absorber present, cadmium present.
3. boron absorber present, no cadmium present.
4. boron absorber present, cadmium present.

Comparing measurements 1 and 3, one finds what fraction of the thermal and higher energy neutrons have been absorbed by the boron.

Comparing measurements 2 and 4, one finds what fraction of the higher energy neutrons have been absorbed by the boron (the thermal neutrons are stopped by the Cd box).

Comparing the difference of 1 and 2 with the difference of 3 and 4, one finds what fraction of the thermal neutrons have been absorbed by the boron.

We plot in Fig. 5 the quantity log A_x, that is, the logarithm of the intensity observed in the detector foil as a function of the thickness of the boron absorber. The lower one of the two straight lines in the figure



is obtained by taking the ratio of measurements 4 and 2. This curve, therefore, refers to high energy neutrons. The higher line is obtained by dividing (3 - r) by (1 - 2). This curve refers therefore to thermal neutrons. The intensity variations with boron absorber thickness may be represented by formulae of the type

$$A_x = A_o e^{-Kx} = e^{-no_o x}$$

The slope of the curves gives the absorption coefficient K. The absorption coefficient in boron is proportional to the -1/2 power of the energy of the neutrons. Therefore, one obtains for the ratio of resonance energy to thermal energy the expression

$$E_{th}/E_r = K_r^2/K_{th}^2$$

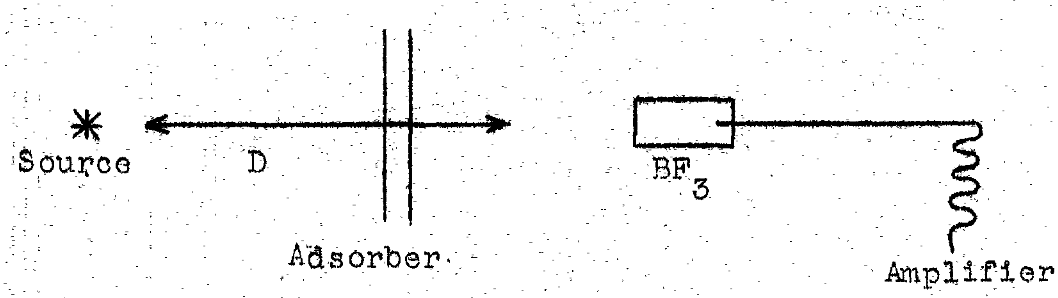
where K_{th} and K_r are the absorption coefficients obtained for thermal energy and for resonance energy. Substituting kT for E_{th}, one can calculate the resonance energy.

One limitation of the method just described is that it assumes that neutrons are removed from the neutron beam only by absorption in boron and not by scattering in boron. This is a valid assumption for small neutron energies where the capture cross-section in boron is large. For energies above a kev, scattering by boron disturbs the measurement.

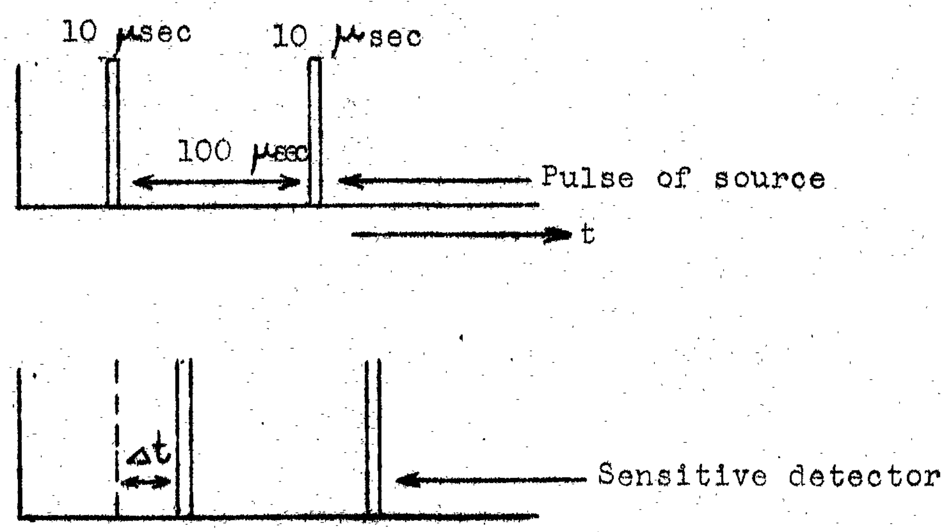
In this manner, the resonance captures for various elements have been explored. For medium and heavy elements it was found that resonances spaced irregularly at a distance of the order of 10 volts from each other are present and that these resonances have widths of the order of 1 volt. In light elements no such resonances are found. These elements should show much more broad resonances which are much more widely spaced, if in fact they do not overlap.

It is more difficult to study the resonances of an isotope which does not become radioactive when it captures a neutron but transforms into another stable isotope. A rather inadequate method which has been applied in such cases is based on the comparison of absorption coefficients of this isotope and of boron.

The velocity selector method of studying slow neutrons may be greatly improved by using an electrical selector rather than a mechanical one. This method is known as the method of modulated neutrons. The experimental arrangement is shown in Fig. 6.



The neutrons originating from the source fall on a neutron absorber and then after having flown a total distance D are detected in a neutron counter, for instance in BF_3 chamber. The source is not emitting neutrons all the time but only during short intervals of time, let us say of 10 microseconds duration, separated by longer time intervals, of for instance 100 microseconds. The neutron pulses emitted by the source are shown in the upper half of Fig. 7.



The lower half of this figure shows the time intervals during which the neutron detector is sensitive. It is seen that the detector will not react to neutrons unless they arrive t after they have been emitted. Thus only neutrons with a velocity

$$v = D/\Delta t$$

are observed. By varying the thickness of the absorber, one can find the absorption coefficient for those particular neutrons in the material of the absorber. By varying the time delay Δt one can investigate the dependence of σ_0 upon neutron energy.

One difficulty that arises in this method is that the neutrons may not be counted in the sensitive time of the detector immediately following the emission pulse of the source but in the sensitive time of the detector which comes 1 period later. This is called recycling. This difficulty can be minimized by choosing a long repeat period for the emission pulses and the corresponding sensitive intervals.

By the method described above it was possible to explore the properties of slow neutrons up to about 500 eV.

We shall no summarize briefly the interaction of neutrons with some elements. This will be done in connection with a table which in part has been taken from H. A. Bethe, Rev. Mod. Phys. 9, 151 (1937).

Isotope	Main Process	Thermal cross section in barns	Resonance cross section in barns	Fast cross section in barns	Remarks
H^1	Elastic soattering	20		1.7	Efficient neutron slow-up short diffusion of thermal neutrons
H^2	Elastic scattering	4		1.7	Good neutron energy degrader with no capture
Li^6	Neutron absorbed alpha emitted	900		1 - 2	Good neutron shield and detector
B^{10}	" "	3000		1.5	" " "
C^{12}	Elastic Scattering	5			No capture
N^{14}	Elastic scattering and neutron absorbed proton emitted	10			
Fe	Elastic scattering and radiative capture	12			At least 1 γ-ray emitted per capture. Fe disturbs ionization chamber measurements by its γ-rays
Rh	Radiative capture	125	6000		Produces active isotope
Cd	" "	2500			Produces stable isotope

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December 14, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Third Series: Neutron Physics Lecturer: J.H. Williams

LECTURE XXIII: FAST NEUTRONS

Generally speaking, neutrons are produced as fast neutrons. There are three principal reactions in which fast neutrons are produced. In the first one an alpha particle enters the nucleus and a neutron is emitted. The reaction is symbolized by (a,n). This reaction has mostly historical interest since it was the first reaction in which the production of neutrons was observed.

A second type of reaction in which neutrons are produced is the reaction in which a deuteron impinges on the nucleus and a neutron is emitted. The symbol for this reaction is (d,n). Two reactions of this type have become of great practical importance. One is the Be(d,n) reaction in which the Be is the bombarded nucleus and which gives a very copious source of neutrons. Another is the D(d,n) in which the deuteron hits a stationary deuteron in the target. In this reaction monoenergetic neutrons can be produced. The energy of the neutrons depends on the energy of the impinging deuteron. Neutrons between 2 Mev and 7 Mev may be produced in this manner if the incident deuterons have an energy of 3.5 Mev.

The third important source of neutrons is the (p,n) reactions and in particular the Li(p,n). In this reaction an Li nucleus is bombarded by protons and neutrons emerge leaving the Be nucleus behind. This reaction is endoenergetic, that is, the protons must have a certain minimum energy in order that the reaction should proceed at all. Monoenergetic neutrons between 10 kilovolts and 2 million volts may be produced by this reaction for the similar range of incident particle energies, i.e., up to 3.5 Mev.

The Li(p,n) reaction may be written in somewhat more detail in the following form

$$Li^7 + H^1 \longrightarrow n^1 + Be^7 + Q$$

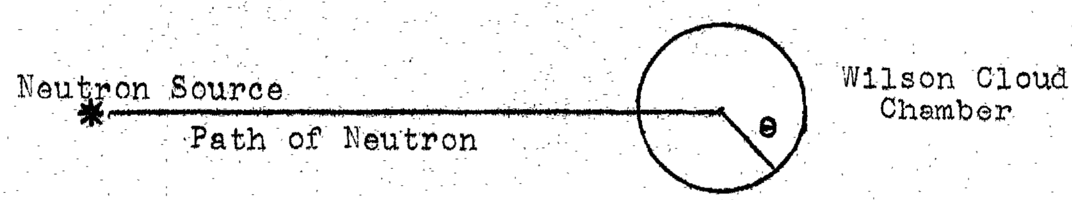
where the energy of reaction Q is a negative quantity, one might suspect that instead of the above reaction the following reaction takes place

$$Li^7 + H^1 \longrightarrow n + Be^{7*} + Q'$$

where Be^{7*} is an excited form of the Be nucleus and Q' is a greater negative reaction energy. If this second reaction proceeds, the neutron produced would be no longer monoenergetic. No evidence of this possibility has been found. It is true that the Li^7 nucleus (which is quite analogous to the Be nucleus having as many neutrons as the Be^7 has protons and as many protons as the Be^7 has neutrons) has an excited state of 485 kilovolts. There is some reason to suspect that the Be^7 has a similar excited state and the presence of such a state might give rise to slower neutrons whenever the Li(p,n) reaction is used to produce neutrons above 1/2 million volts.

Energy Measurement of Fast Neutrons

The first method of measuring the energy of fast neutrons made use of the Wilson Cloud chamber. The arrangement by which this measurement may be performed is shown in the following figure:



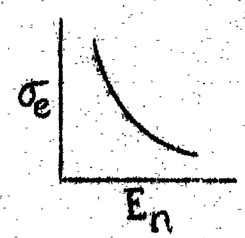
On the left hand side the position of the neutron source is indicated. On the right hand side, a Wilson Chamber is schematically shown. The chamber is filled with a gas of small atomic weight such as hydrogen or helium. The path of the neutron is also indicated in the figure. The heavy line in the chamber is the track of a recoiling particle. If this track makes the angle θ with the continuation with the path of the neutron then from this angle and from the energy of the recoiling particle, the original energy of the neutron may be calculated. In particular, if the recoiling particle is a proton then the energy of the neutron is given by the equation:

$$E_r = E_n \cos^2 \theta$$

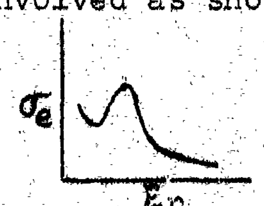
Where E_r is the energy of the recoiling particle and E_n is the original energy of the neutron, it is usual to count only the tracks which go in the forward direction and make a small angle θ (for instance, smaller than 10°) with the direction of the incident neutron. Then the factor $\cos^2 \theta$ in the above formula may be neglected and the energy of the recoiling proton is equal to the energy of the incident neutron.

One danger of this procedure is that the neutron might have been scattered in the wall of the chamber and while the proton seems to go forward, it actually does not lie in the continuation of the path of the neutron which has hit it. Another difficulty is that the tracks of the recoiling particle are rather long and they may not end in the chamber. The energy of such recoils cannot be measured.

A further difficulty in evaluating results of cloud chamber measurements as well as of other measurements of fast neutrons is that a previous knowledge of the dependence of the collision cross section on the energy of the incident neutron is required. A typical dependence of this cross section σ_e (which stands for elastic cross section) on the neutron energy E_n is shown in the following figure:



Unfortunately sometimes, as for instance, in the case of helium this dependence is more involved as shown in the figure:



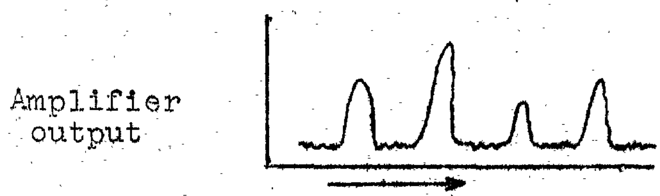
in which a resonance is seen between 1 and 2 Mev. This resonance gives rise to a great number of tracks if the incident neutron has an energy between 1 and 2 million volts. Previous to the discovery of this resonance these many tracks were erroneously interpreted as showing the presence of many neutrons between 1 and 2 Mev.

Recently photographic plates have been used instead of cloud chambers in measuring the energy of neutrons. The emulsion of the photographic plates contains hydrogen. The recoil protons cause ionization and leave developable grains in the wake which line up in the path of the proton. Again as in the case of the

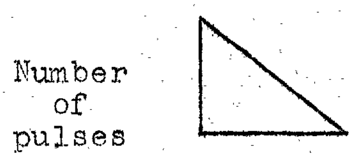
Wilson Chamber only those tracks are to be counted which make a small angle with the direction of the neutron. The photographic technique has the following advantages as compared to the Wilson chamber. First, the stopping power of the emulsion is greater, the length of the tracks is only a few thousandths of an inch and the danger that the track leaves the emulsion before ending is smaller. Second, in the photographic plate there need be much less material than there is in a Wilson chamber; consequently the danger of a scattered neutron causing a recoil is decreased. Thirdly, the photographic plate is continuously sensitive while the sensitivity of a cloud chamber is restricted to the moments of expansion. The photographic technique has however the disadvantage that it becomes inaccurate in counting slower neutrons. The recoils produced by such neutrons make few grains and since the number of the grains produced is used in determining the neutron energy, fluctuations in the number of grains make such energy determinations difficult. The photographic method must be calibrated by using a reaction of known energy.

A less straightforward, but perhaps more trustworthy method of measuring neutron energies makes use of the ionization chambers. Such a chamber is shown schematically in the following figure:

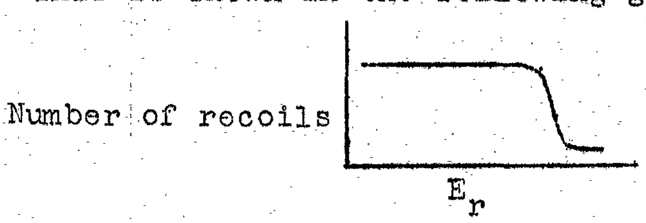
The neutron source is seen on the left hand side. The chamber is filled with hydrogen. The field within the chamber collects the ions produced by the recoiling protons. On the right hand side the connection to the amplifying grid is shown in which the charge produced by the ions is amplified so as to give a measurable effect. The following figure shows these amplified current pulses, each of which corresponds to one recoiling particle according to the angle which the recoiling proton makes with the neutron. The recoiling particle will carry more or less energy and the size of the pulse will vary correspondingly.



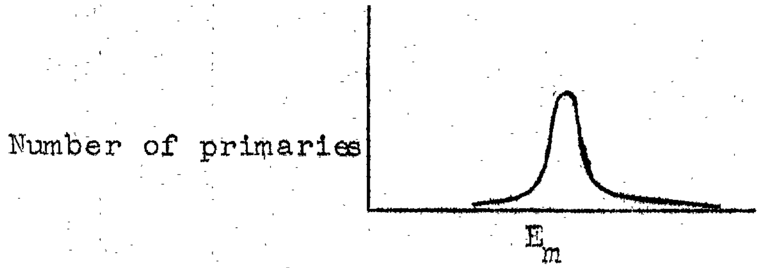
One may plot the number of pulses which have more energy than a given value E against amplifier output pulse size or energy.



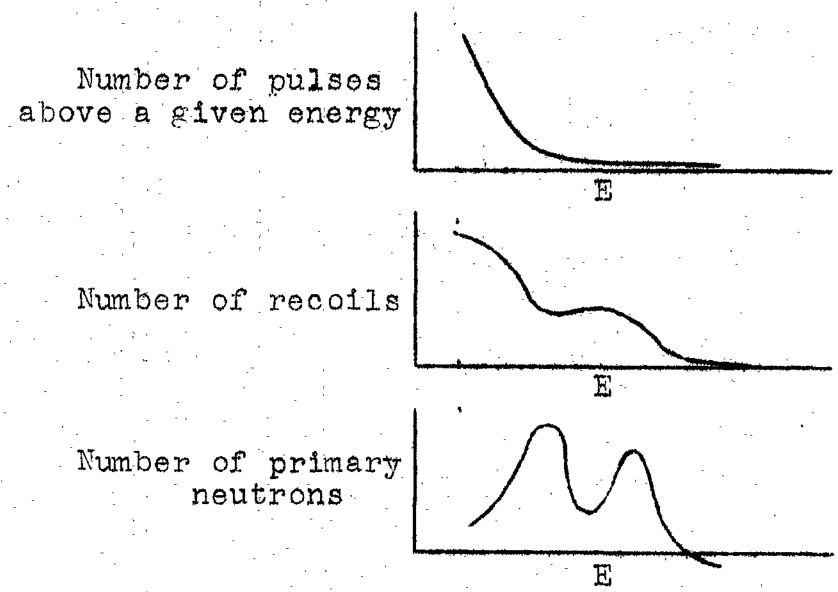
Very small pulses cannot be counted because they are obscured by the background or noise of the amplifier. By differentiating this curve, one obtains the number of recoiling particles of a given energy. This is shown in the following graph:



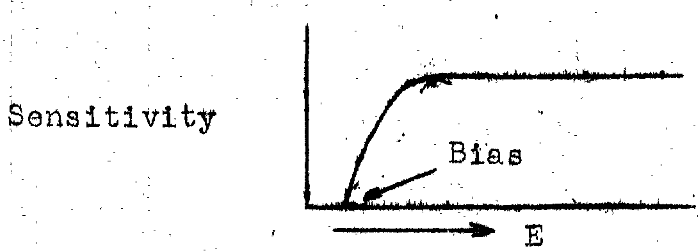
If the gas in the ionization chamber was hydrogen then a second differentiation gives the number of primary neutrons as plotted against their energy.



The figures given above refer to the case of monoenergetic incident neutrons. If a distribution of neutrons with various energies impinges on the ionization chamber, figures of the following kind may be obtained:

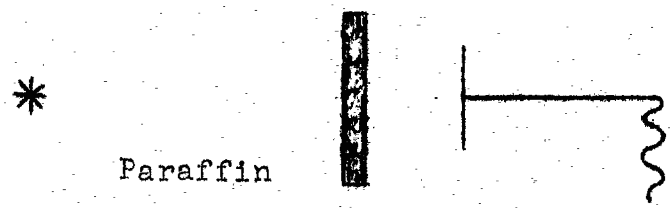


Instead of recording the pulses of various height, one might use an amplifier with a bias such as to cut out all pulses less than a certain energy. The sensitivity of such a counter is shown below:

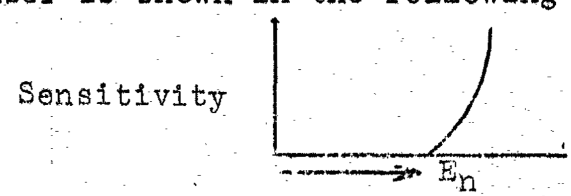


Below the bias the amplifier-detector will not respond at all. At higher energies the sensitivity will rise to a plateau and then decrease.

Instead of using an ionization chamber filled with hydrogen gas, one might use a chamber in which the side facing the neutron source is coated with a thick layer of paraffin.

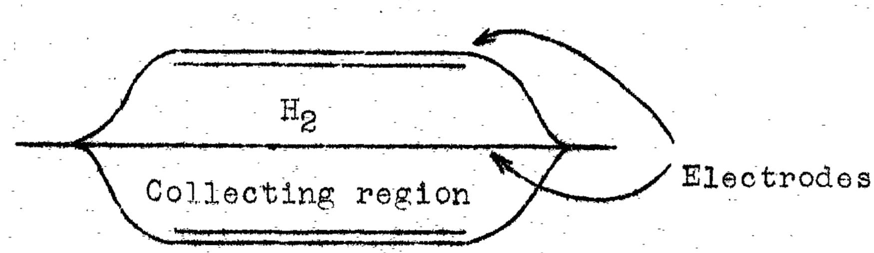


Such chambers are then filled with argon. The argon nuclei are sufficiently heavy so that argon recoils carry relatively little energy and the bias may be adjusted in such a way as to count only the proton recoils originating in the paraffin. The sensitivity of such a chamber is shown in the following figure:



There is this time an increasing rise of sensitivity above the bias value. This is due to the fact that higher energy neutrons produce protons which have a longer range in the paraffin and have therefore a bigger chance to get into the ionization chamber.

Electric amplification may be partly replaced by so-called gas amplification. This is done in proportional counters. Such a counter is shown schematically in the following figure:

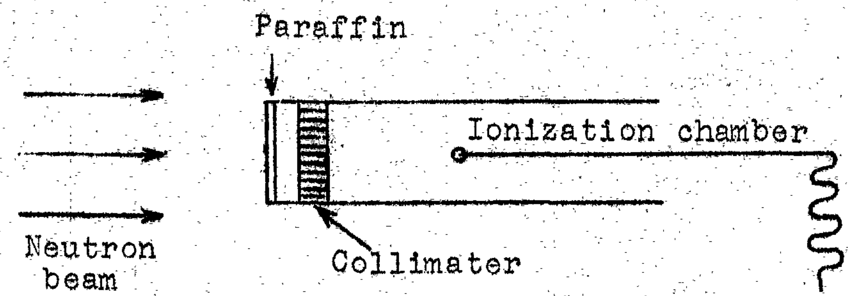


The counter is filled with hydrogen. Instead of a collecting plate it has a collecting wire near which large electric fields are present. In this field electrons are accelerated towards the wire causing further ionization and in this way greater pulse result. With such proportional counters, pulses as small as 10 kv may be detected while the ionization chamber described above is insensitive up to 100 kv or higher.

Ionization chambers also may be filled with BF_3. In such chambers the neutron reacts with the B^{10} nucleus giving rise to end products of total kinetic energy about 3 Mev. Thus sufficiently big pulses are produced.

Not only the measurement of the energy of fast neutrons but also the measurement of their number is a difficult question. In one method due to Fermi and Amaldi, the neutrons are slowed down in a water bath of sufficiently big volume surrounding the neutron source on all sides. After being slowed down, the neutrons are detected by rhodium or indium foils distributed in the bath. From the known cross sections of these foils for slow neutrons, one may determine the original number of fast neutrons. In order to do that one has to integrate over the various positions of the detectors in the bath. This last operation may be avoided if instead of the foils the slow neutrons are detected by manganese dissolved in the bath. By stirring this solution and by taking a sample afterwards, one may find the integral of neutron absorption in the bath and from this quantity the original number of fast neutrons may be calculated.

A theoretically very effective way of counting fast neutrons and measuring their energy is shown in the following sketch:



On the left hand side, an incident neutron beam is shown. This beam impinges on an ionization chamber which on the side of the neutron beam has a thin paraffin film. This film is followed by a collimator which is a plate transversed by narrower channels pointing in the original direction of the neutron. These channels will let through only such protons whose direction coincides with the original direction of the neutrons. From the number of protons in the thin paraffin layer, from the collision cross section between protons and neutrons and from the number of pulses of various sizes in the ionization chamber the number of neutrons with various energies in the incident beam may be determined.

Unfortunately, the arrangement does not work. The reason is that there is too much hydrogen in other places than in the thin paraffin layer and consequently not all recoil protons come from that paraffin layer.

Fast Neutron Processes

We shall discuss the following processes which occur when fast neutrons impinge on nuclei. Elastic scattering, inelastic scattering, capture, (n,α) process, (n,p) process, (n,2n) process. A cross section can be assigned to each of these processes. These cross sections are designated (σ_o, σ_i, σ_c, $\sigma_{(m,a)}$, $\sigma_{(n,p)}$, $\sigma_{(n,2n)}$.

One simple experiment which gives information about these cross sections is to place a small scatterer between the neutron sources and the neutron detector. If the scatterer is small enough and if the neutron has been deflected in the scattering process by a big enough angle, then this neutron in the original beam will miss the detector and no neutron outside the original beam will have an appreciable chance to be scattered into the detector. In this way the major portion of the elastically scattered neutrons will be missing from the beam. Similarly the inelastically scattered neutrons will be only partially observed. In the case of the inelastically scattered neutrons an argument may be given which shows that these neutrons are distributed spherically after the scattering process. In fact the scattering process may be described by the following equation:

$$n^1 + Z^A \longrightarrow Z^{(A+1)*} \longrightarrow Z^{A*} + n^1$$

This equation shows that the first act in an inelastic scattering process is the formation of a compound nucleus with the atomic number $A + 1$. When this compound nucleus emits the neutron, the neutron will have forgotten about its original direction of incidence. Thus there is no reason for inelastically scattered neutrons to be scattered over particularly small angles and the scattering geometry described above will cause most inelastically scattered neutrons to miss the detector. The neutrons which are captured (with gamma emission) or which undergo a (n,α), or (n,p) reaction will be naturally missing from the original beam. In the $(n,2n)$ process the neutrons emitted are again spherically distributed and will therefore in all probability miss the detector. Thus practically whatever reaction a neutron has made with the scattering material the result will be that the neutron will not appear in the detector. The intensity I transmitted by the scatterer will be related to the intensity I_0 of the original beam by the formula:

$$I/I_0 = e^{-nx\sigma}$$

Here small n is the number of nuclei per cubic centimeter in the scatterer, x is the thickness of the scatterer and σ is the sum of all cross sections or the total cross section.

LA-24 (24)

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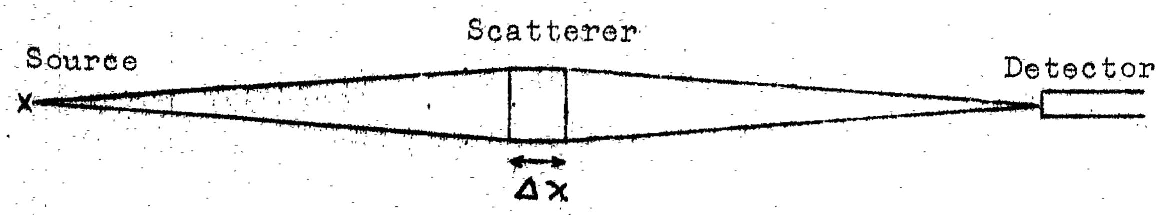
LECTURE SERIES ON NUCLEAR PHYSICS

Third Series: Neutron Physics Lecturer: J.H. Williams

LECTURE XXIV: PROPERTIES OF FAST NEUTRONS. CROSS SECTION MEASUREMENTS.

The total cross section of a substance for fast neutrons can be measured by a direct transmission experiment in which the source, scatterer, and detector are situated in what is known as a "good geometry". This good geometry is so defined that the solid angle subtended by the scatterer at either the position of the source or of the detector is small.



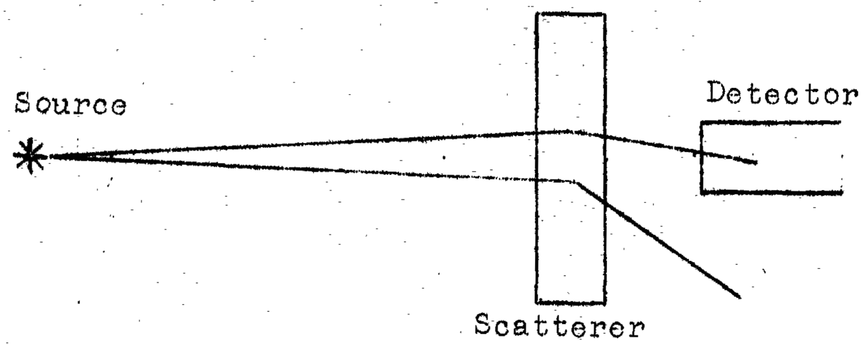
GOOD GEOMETRY

In this case the neutrons from the source strike the scatterer at practically normal incidence and any process undergone by a neutron in its passage through the scatterer will result in its removal from the beam and its consequent failure to be recorded by the detector. The neutrons counted by the detector will consist almost entirely of those which have had no interaction with the atoms of the scatterer. A very small percentage of the detected neutrons will consist of those scattered (elastically or inelastically) through very small angles, but in a really good geometry they will constitute a negligible fraction. A transmission experiment in such a good geometry will therefore measure the total cross section for all processes. If the incident neutrons are normal to the scatterer, which may be supposed to have a thickness Δx and a number n atoms per cm^3, the ratio of the detected intensities with and without scatterer will be given by

$$I/I_0 = e^{-n\sigma\Delta x}$$

where σ is the total cross section.

In contrast to this measurement with good geometry, transmission experiments in poor geometry are performed in order to find the elastic scattering cross section. In this case the solid angle subtended by the scatterer at the position of the detector is very large. This makes it possible for neutrons scattered through large angles to be scattered into the detector, thus compensating



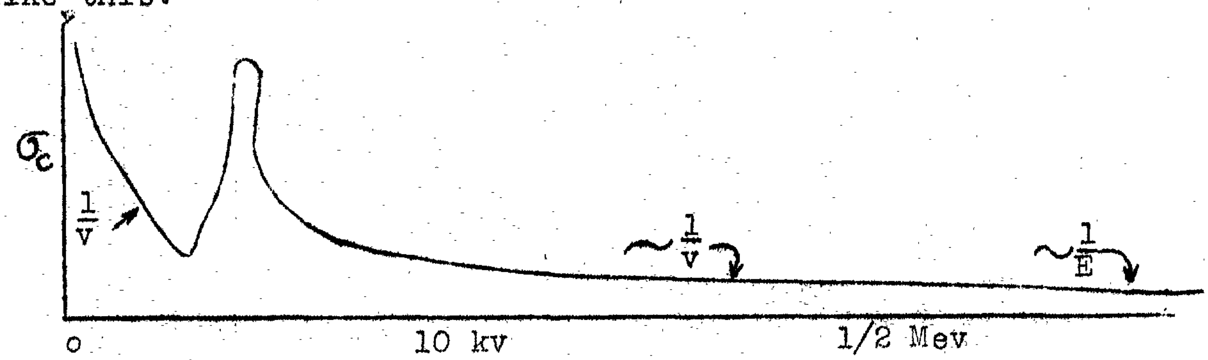
POOR GEOMETRY

for those scattered neutrons which miss the detector. Since this compensation is almost exact, scattering processes alone would have no net effect on the intensity recorded by the detector. However, if the detector is biased so as not to record neutrons below a certain energy, the neutrons which are <u>inelastically</u> scattered will still contribute to the measured reductions in beam intensity along with all the other processes with the exception of <u>elastic</u> scattering.

This experiment with poor geometry and biased detector therefore measures the sum of all cross sections with the exception of the cross section for elastic scattering. To find the elastic scattering cross section alone, we therefore subtract the cross section as measured in poor geometry with a biased detector from that measured in good geometry.

Capture cross sections are easily measured if the nucleus resulting from the addition of the neutron is radioactive. The cross section can then be obtained from a measurement of the induced activity. When the resulting nucleus is not radioactive there is no easy means for measuring the capture cross section. It sometimes happens that one isotope of a given element will become radioactive upon capture of a neutron while some other isotope of the same element will not. However, it is found empirically that the capture cross sections of isotopes of a given element are of the same order of magnitude. Therefore one can sometimes estimate the capture cross section of an isotope which does not give a radioactive product.

Dividing the energy range into roughly three regions, the resonance region from 0 to 10 kv, the medium fast region from 10 kv to 1/2 Mev, and the region of fast neutrons from 1/2 Mev on up, one gets a behavior of the capture cross section with energy which looks like this:



In the resonance region there are sharp resonance peaks in the cross section superposed on a 1/v background. In the medium fast region the resonances are much broader and are discrete only for the lighter elements, say up to oxygen. For the heavier elements the broad levels overlap and give a smooth behavior something like 1/v. In the fast region above 1/2 Mev the behavior is more like 1/E. For the very fast neutrons the cross section is determined principally by the area of the nucleus and the competition with other possible processes like inelastic scattering which becomes more probable as the energy of the neutrons increases.

The cross section for an (n,α) or an (n,p) reaction can be measured by putting a sample of the element in question inside an ionization chamber which will count the alpha particles or the protons directly, when the chamber is placed in a neutron flux. The cross section for an $(n,2n)$ reaction is extremely difficult to measure except in the case that the residual nucleus is radioactive.

Inelastic scattering of a neutron is characterized by a reduction in the energy of the neutron as a result of the scattering process. In light elements like hydrogen or helium or carbon this reduction in energy comes from the normal classical collision of two particles of comparable mass and such collisions should really be called elastic. However for the heavier elements like lead or gold, the neutron enters the nucleus and emerges with reduced energy leaving the residual nucleus in an excited state. The excited nucleus then falls down into the ground state with the emission of a γ-ray. Thus a nuclear inelastic scattering is always accompanied by one or more γ-rays. The probability for inelastic scattering increases with increasing atomic weight since the heavier nuclei, having many constituent particles, have a large variety of closely spaced energy levels into which the nucleus can be raised by the addition of the energy of the incident neutron.

Qualitative information about inelastic scattering can be obtained by measuring the activity induced by the inelastically scattered neutrons in various detectors which respond to neutrons in different energy ranges. For example, silver or rhodium can be used as radioactive detectors of slow neutrons and aluminum or silicen as detectors of the fast neutrons. By measuring the activities in these detectors with and without the scatterer one can get some idea of how the inelastically scattered neutrons have to be degraded in energy. Experiments have also been performed in which the γ-ray intensity which accompanies any inelastic scattering is measured. The magnitude of this intensity gives an indication of the cross section for the inelastic scattering. However the experiment is complicated by the possible presence of γ-rays in the source. Not much information about the energy of the inelastically scattered neutrons can be obtained from the energy of the γ-rays since the excited nucleus will in general emit a variety of γ-rays of different energy corresponding to the many states into which the nucleus can fall before reaching the ground state. Complete information would be given by a measurement of the spectrum of the inelastically scattered neutrons but this is a very difficult thing to do.

BIOLOGICAL EFFECTS OF RADIATION. PROTECTION.

Electromagnetic radiation like x-rays or γ-rays damage tissue by producing fast electrons (through the photoelectric effect or the Compton effect) which cause ionization and disruption of the molecular structure of the tissue. These electrons have a relatively small ionization per unit path length and a correspondingly long range. Neutrons will in many cases give rise to energetic heavy charged particles like protons or α-particles which will have a very high specific ionization but a short range. Fast protons could be produced by collision of a fast neutron with a hydrogen atom in one of the molecules. Slow neutrons may be captured and give rise to γ-rays or in some cases, like those of boron or lithium or nitrogen give α-particles or protons through (n,α) and (n,p) reactions.

Against γ-rays, lead or concrete walls of sufficient thickness give good protection. Lead is of no value against neutrons, however. Materials containing hydrogen, like water, paraffin, or concrete, are generally used because of their effectiveness in slowing the neutrons down to a point where they can be captured. The γ-rays resulting from the capture can be absorbed by using lead sheets.

December 21, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Fourth Series: Two-Body Problem Lecturer: C. L. Critchfield

LECTURE XXV: NUCLEAR CONSTANTS

There are two kinds of nuclear quantities: quantized and not quantized. The quantized numbers are: mass number, charge, spin, statistics and parity. The quantities that are not quantized are the nuclear radius, the nuclear mass, the magnetic moment and the electrical quadruple moment.

It is believed that eventually there will exist a theory of the quantized nuclear numbers correlating these to each other. Such a theory shall link the various numbers to each other and possibly to other physical quantities. Up to now there has been only one significant attempt in this direction. Pauli tried to correlate spin and statistics by stating that particles with a half integral spin have Fermi-Dirac statistics while particles with an integer spin have Bose-Einstein statistics.

The meaning of Fermi-Dirac statistics may be explained with the help of the example of the H_2 molecule. Let us assume that the spins the two protons of this molecule (which have the value 1/2) are parallel (triplet state). Then Fermi-Dirac statistics, which applies to the protons, postulates that an interchange of the positions of the protons will cause the wave functions to change sign.

The consequence is that a rotational state with no node ($J = 0$) does not occur simultaneously. All even values of the rotational quantum number J are excluded. Odd J values are allowed and the lowest state corresponds to $J = 1$. This state of hydrogen is called orthohydrogen.

Let us now consider the D_2 molecule. Let us again assume that the spin of the deutrons (which have the value 1) are parallel. Then the Bose-Einstein statistics which applies to the deuterons postulates that the even rotational states of the molecule are alone present while the odd rotational states do not occur. This state of the deuteron molecule is called orthodeuteron.

There also exists a second state for both the hydrogen and the deuteron in which the spins of the nuclei are not parallel and in which those rotational states are allowed which did not occur in the ortho states and those rotational states are missing which were present in the ortho-molecules. These states are called parahydrogen and paradeuterium. The experience about the statistics of nuclei is a strong argument in favor of the nuclei being constituted from neutrons and protons rather than from protons and electrons. Assuming Fermi-Dirac statistics for all these elementary particles (protons, electrons and neutrons) it follows that a nucleus containing an odd number of such particles behaves according to the Fermi-Dirac statistics while a nucleus containing an even number of particles has Bose-Einstein statistics. Now the deuteron, if it were constituted from protons and electrons, would have to contain two protons and one electron, that is, an odd number of particles and should, therefore, have

Fermi-Dirac statistics, whereas in reality it has Bose-Einstein statistics. If on the other hand, protons and neutrons are the constituents of the nuclei, a deuteron is made up of one proton and one neutron, that, of an even number of particles which clearly predicts Bose-Einstein statistics.

Among the non-quantized quantities characteristic of nuclei, the radius can not be determined very accurately. Approximate values have been obtained at an early date in the history of nuclear physics by scattering of alpha particles. The principle of this determination is that when the alpha particle approaches to within the sum of the radii of the scattering nuclei and the alpha particle then the Rutherford scattering law ceases to hold. From this anomalous scattering of alpha particles nuclei radii in the order of 10^{-12} cm have been obtained.

The electric quadruple moment of a nucleus for a sharply defined quantity has not been determined up to now with very high precision.

It has been first discovered and measured by investigating the hyperfine structure of atomic spectrum. This hyperfine structure is due to the interaction of the magnetic moment of the electron (which may be due to the spin or the orbital moment of the electron with the magnetic moment of the nucleus. Different relative orientations of the nuclear and electrical angular moments produce states of slightly different energies and cause a narrow splitting of the spectral lines which is as a rule less than one wave number. The magnetic interaction described above gives a certain regularity in the hyperfine structure. Small deviations from this regularity were discovered in heavy nuclei. These deviations could be explained by an additional term in the energy explanation which depends on the square of the cosine of the angle included between directions of the nuclear and electric angular moments. Physically this additional term is due to an electric quadruple moment of the nucleus which interacts with the inhomogeneity of the electric field produced by the electrons. There is also one rather direct determination of the quadruple moment of the deuteron which we shall discuss later.

The masses of the nuclei are known quite accurately from mass spectrographic work. This method uses crossed electric and magnetic fields which focus ions of a given weight into a given position. Recent accurate determinations of masses by this method have been carried out be Bainbridge for light nuclei and by Dempster for heavy nuclei. Bainbridge was able to obtain high intensities by focusing a beam of great angular deviations. In his set-up the position of the focus depends on the mass of the nucleus very accurately as a linear function. In this way it was possible to obtain mass determinations of high accuracy by comparing the focus of ions of nearly the same weight. Thus in his early work he compared the mass of the helium ion with the mass of the tri-atomic DHH. The actual value deduced from this measurement for the mass of the deuteron turned out to be erroneous because it was based on the pre-1935 ratio of helium to oxygen masses given by Aston. More recent work of Bainbridge carried out with abudant deuterium sources and evaluated with the correct helium to oxygen ratio gave very accurate deuteron masses.

Nuclear masses are also used in balancing reaction energies. A systematic study of these reaction energies lead Bethe and Rutherford in 1935 independent from each other to recognize that the earlier measurements in the helium to oxygen ratio there must have been an error.

The following conventions on nuclear masses should be remembered. In the units used by physicists O^{16} has exactly the mass 16. In the units which the chemists use the natural isotopic mixture of oxygen has exactly the atomic weight 16. In every case the masses of neutral atoms rather than the masses of nuclei are given.

Determinations of nuclear masses are closely connected with the history of the discovery of deuterium. At first the chemical and the mass spectrographic determination of the mass of hydrogen atom seemed to agree; thus indicating that hydrogen has no isotope. With the discovery of an O^{18} isotope however it became necessary to postulate that also hydrogen has a heavier isotope. Birge and Menzel recognized this and following their suggestion Urey and his collaborators searched and found the heavy hydrogen.

A list of the best known values for the masses of the light atoms is given in the following table. The last decimal place in this table is uncertain.

n	1.00893	H^3	3.01704
H	1.00812	He^3	3.01701
D^2	2.01472	He^4	4.00388

From the last values one can draw conclusions concerning the binding energy of the nuclei. Thus the binding energy of the deuteron is 2.16 Mev while the binding energy of the helium nucleus is 28 Mev.

It is surprising that by merely doubling the number of particles within the nucleus the binding energy should be increased by a factor of 13.

The magnetic moment is also known with very high accuracy. The best determinations were carried out at Columbia University by Rabi and his collaborators. They used an ingenious method. The measurements were carried out on molecules like H_2 or D_2 in which the electrons are paired and the effect of the electron spins cancel each other. A beam of the molecule passes consequently through 3 magnetic fields. The first of these fields is inhomogenous and deflects the molecular beam. The second field is homogenous. Around this field the nuclear spins perform a processing motion. The third field is inhomogenous again and brings the molecular beam back to a focus in which the molecules meet independent of the spin orientation of the nuclei. In addition to this arrangement an oscillating magnetic field is superposed on the homogenous magnetic field. This oscillating field imparts energy to the nuclei causing the nuclear spin to jump from one state into another. If this happens the second inhomogenous field will no longer compensate the deflection caused by the first inhomogenous field and any change in orientation caused by the oscillating field will give rise to a diminished number of molecules arriving in the focus. A change in the nuclear spin orientation will occur however only if the frequency of the oscillating magnetic field is in resonance with the precession of the nuclear spin. This precession depends on the strength of the homogenous field, on the magnetic moment and on the value of the nuclear spin. The last quantity may be determined from other experiments, for instance by the analysis of the spectrum.

In this way then, the magnetic moment may be determined to a high accuracy. Actually the precision of the experiments is limited only by the length of the region of the homogenous electric field in that even the time of flight of the molecule in this region is short. The accuracy of resonance between an oscillating magnetic field and precession frequently is limited. Very accurate values have been obtained by the relative magnitudes of magnetic moments and the absolute values are chiefly uncertain due to difficulties in accurately measuring the magnitude of the homogenous magnetic field.

For the simplest measurement of the magnetic measurements of proton and deutron the molecule HD is most appropriate. If the molecular beam is formed at a sufficiently low temperature and the molecules are present in the 0 rotational state, the minima of the beam intensity in the focus will correspond to precession frequencies of the H and D nucleus.

A more complex picture is obtained if the H_2 molecule is used. Here the 0 rotational state can not be used because in this state the nuclear spins have opposite directions and the magnetic moments compensate. Therefore the first rotational state ($J = 1$) is to be used. In this state also the spin of the nuclei add up to one and there will result altogether 9 rotational and spin states. The energies of these states are affected by the magnetic interaction between the protons and by the motion of the electrons induced by molecular rotation. Between the 9 states mentioned above, two sets of 6 transitions are possible and correspondingly one observes two sets of 6 dips of intensity. In order to explain the exact frequency at which these dips occur, assumptions have to be made about the interaction between the magnetic moments of the nuclei and the magnetic field produced by the rotating molecule. Giving appropriate values to these conditions the position of the 6 dips can be explained.

Having determined the molecular quantities that have an effect on the nuclear magnets the position of the dips in D_2 may be predicted. For this molecule one may investigate the state in which J = 1 and also the spin is equal to 1. The predictions however did not check with the experimental results and agreement could be obtained only by assuming an additional term in the energy which depends on the square of the cosine included by the nuclear moment and the molecular axis. This is the kind of term that should be if the deuteron had a quadruple moment. Rabi, Ramsey and others have indeed been led to assume an electric quadruple moment for the deuterium.

The discovery of the quadruple moment of the deuterium came as a surprise because calculations on the binding energy and other properties of the deuteron were based on a simple symmetric model of the deuteron which did not permit the existence of any quadruple moment. The presence of such a moment shows that even in the simplest two-body problem of nuclear physics quadruple forces appear. The following table gives the magnetic moments of the simplest nuclei measured in units of nuclear magnitrons.

H^1		2.785
n^1	-1.93	
D^2		.855

The negative side attached to the magnetic moment of the neutron is necessary in order that the proton and the neutron magnets lined up in a parallel way should give at least approximately the magnetic moment of the deuteron.

It is remarkable that the magnetic moments seem to be rather closely additive. The algebraic sum of the proton and neutron moments is closely equal to the moment of the deuteron. This should be expected offhand only if the deuteron had a simple symmetric structure and the presence of a quadruple moment shows that there is some deviation from symmetry.

The magnetic moment of the neutron has been measured by Bloch and Alvarez. The principle of the measurement is similar to that used by Rabi and his collaborators but instead of the first and last inhomogenous magnetic fields, they used two ferromagnetic sheets. It has been shown by Bloch that the scattering of neutrons by ferro-magnets depends on the orientation of the spin relative to the magnetic field in the ferro-magnet, therefore the ferro-magnetic sheets can be used as polarizers and analyzers between the two sheets. The neutrons pass through a homogenous magnetic field on to which an oscillating field is superposed. This part of the outfit acts in the same way as the corresponging part in the Columbia measurements and resonances manifest themselves again by changes in intensity of the beam in the focus.

December 23, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Fourth Series: Two Body Problem Lecturer: L. C. Critchfield

LECTURE XXVI: NEUTRON-PROTON INTERACTION

The earliest attempts to build a more detailed theory of nuclear forces were concerned with explaining the binding energies of the light nuclei, particularly of the deuteron. It was noted that the binding energy of the helium nucleus was thirteen times that of the deuteron although the number of attractive links between four particles is only six, at most, compared with one between two particles. Wigner pointed out that the extraordinarily large binding in the alpha particle compared with that of the deuteron indicated that the kinetic energy of the particles in the deuteron almost compensates the mutual attraction and that this can be the case if the attractive forces have a very short range. i.e., the range of forces is smaller than the wave length of the particles. In the alpha particle, on the other hand, the number of attractive bonds is proportionately greater than the increase in kinetic energy, and a lower average energy is possible.

Short range forces between nuclear particles makes possible many simplifications in the theory of the deuteron. Essentially, we may say that if neutron and proton are farther apart than a distance a, they do not influence each other appreciably, but at separations less than a strong attractive forces act and there is a deep valley in the plot of potential energy against separation. In terms of wave concepts there is a strong refraction of the neutron and proton waves near their center of gravity. Since the width of the standing wave that represents the stable deuteron is large compared with that of the refractive region, however, the exact shape of the region cannot be important to the description of the state. Accordingly, it may be assumed that the potential energy is a constant V_0, inside the radius a and zero outside it.

Let r be the relative coordinate of the proton with respect to the neutron, M the mass of proton or neutron, and E the energy of the system. The wave equation is then, reducing the two body problem to a one body problem in the usual way:

$$\frac{\hbar^2}{M}\nabla^2\psi(r,\theta,\varphi) + (E-V)\psi(r,\theta,\varphi) = 0$$

We wish the solution of lowest energy for which we know $E = -\epsilon =$ -2.16 Mev. Since the attractive potential exists only at small distances we consider only the spherically symmetrical solution (an angular momentum keeps the particles apart, i.e., the wave functions would have a node through $r = 0$).

We accordingly let

$$\psi(r,\theta,\varphi) = \frac{u(r)}{r}$$

and the wave equation becomes

$$\frac{d^2u}{dr^2} = \frac{M}{\hbar^2}(V-E)u$$

Outside $r = a$, $V = 0$ and $u = Ae^{-r\sqrt{M\epsilon/\hbar^2}}$. The solution with positive exponent is excluded by the condition of finite wave functions. Inside $r = a$, $V = -V_0$, a negative energy that is large in absolute value compared with E.

The solution is then,

$$u = B \sin r\sqrt{\frac{+M(V_0-\epsilon)}{\hbar^2}}$$

The cosine is excluded. To join these two solutions smoothly their logarithmic derivatives must be equal:

$$\left(\frac{u'}{u}\right)_{in} = \left(\frac{u'}{u}\right)_{out} \sqrt{\frac{M(V_0-\epsilon)}{\hbar^2}} \quad \cot a\sqrt{\frac{M(V_0-\epsilon)}{\hbar^2}} = \frac{M\epsilon}{\hbar^2}$$

$$a\sqrt{\frac{MV_0}{\hbar^2}} = \arccos\sqrt{\frac{\epsilon}{V_0}} \cong \frac{\pi}{2}$$

whence

$$a^2 V_0 \cong \frac{\pi^2 \hbar^2}{4M}$$

Thus the existence of the deuteron suffices to determine the product of the depth of the potential well and the square of the width of the well. If we express V_0 in units of $+mc^2$ and a in units of e^2 / mc^2 we get $a^2 V_0 \cong 25$

An independent determination of a or V_0 could be made by solving for the wave functions in the three or four body problems, or from scattering experiments. The best indication of the size of a comes from an interpretation of the scattering of protons by protons. It is then assumed that a is the same for the deuteron. In the units chosen the best value of a is apparently about unity:

So far, only the space coordinates of the particles have been considered. It is well established that the ground state of the deuteron is a triplet, i.e., the spins of neutron and proton are parallel. Thus the above rough calculation applies to the triplet state. Furthermore, the wave function assumed is spherically symmetrical so that no quadrupole moment is obtained. In spite of these defects, however, the scattering induced by the forces thus derived will be calculated and compared with experiment.

Scattering by short range forces is particularly easy to compute, especially if the wave-length of the colliding particles is longer than the range of forces. Under these conditions a finite orbital angular momentum will prohibit the particles from coming close enough together to be attracted and there is no scattering if the particles pass each other at a distance larger than a wavelength. Consider a plane wave for the neutron incident upon a proton. This wave represents a definite relative velocity v of "collision" but for all possible values of the impact parameter and hence all angular moment. According to the foregoing argument deflection will be experienced only by those collisions of zero angular momentum, i.e, in the spherically symmetric state. Let the incident beam be represented by $\Psi = e^{ikr\cos\theta}$. The spherical part of Ψ in absence of a potential well can be determined by averaging over all θ.

$$\Psi_o^{\,o} = \frac{\sin kr}{kr}$$

In the presence of a field the wave will be strongly refracted inside $r = \underline{a}$ and, so far as the wave at large distances is concerned, the effect will be to introduce a phase shift δ. The general form of the spherical part with potential well is then

$$\Psi_v^{\,o} = C\,\frac{\sin(kr+\delta)}{kr}$$

with C and δ to be determined. The wave $\Psi_v^{\,o}$ must represent the spherical part of the incident beam plus a scattered wave, and the latter must have the form Se^{ikr}/kr. Hence

$$-\frac{1}{2i}\left\{e^{ikr}-e^{-ikr}-C\left[e^{i(kr+\delta)}-e^{-i(kr+\delta)}\right]\right\}-Se^{ikr}$$

$$C = e^{i\delta} \qquad S - \frac{e^{2i\delta}-1}{2i} = +\,e^{i\delta}\sin\delta$$

The number of scattered neutrons per unit volume is then

$$\frac{\sin^2\delta}{k^2r^2}$$

and the total number of particles scattered by the proton per second is

$$\frac{4\pi}{k^2}\,v\,\sin^2\delta$$

The incident current density postulated is v hence the effective cross-section presented for scattering is

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta$$

Determination of δ can be made approximately as follows: the bombarding energy of the neutrons is assumed to be small compared with depth of the potential well, hence the wave function inside the range of forces, a, is substantially the same as for the stable deuteron. To join functions smoothly, therefore

$$-\sqrt{\frac{\epsilon M}{\hbar^2}} = k\cot(ka+\delta)$$

$$\frac{M\epsilon}{\hbar^2k^2} = \frac{1}{\sin^2(ka+\delta)}-1$$

neglecting ka compared with δ and calling ϵ, the absolute value of the binding energy of the deuteron

$$\sin^2\delta = \frac{1}{1+\epsilon/E} \qquad \sigma = \frac{4\pi\hbar^2}{M(\epsilon+E)}$$

Thus the binding energy of the deuteron determines the scattering cross-section, at least if the collision is sufficiently slow. Comparison with experiment, however, showed that the predicted cross-section is a little high at high energy ($E > 2$Mv) and about a factor 10 too small at low energy. To explain the discrepancy Wigner pointed out this cross-section is calculated from what is known about the triplet state of the deuteron only and that it is possible to choose a "binding" energy, ϵ', for the singlet state in such a way as to account for the experimental results. Since three out of four collisions between neutron and proton are triplet collisions and one is singlet the result for the complete cross-section is

$$\sigma = \frac{4\pi\hbar^2}{M}\left\{\frac{3}{4}\,\frac{1}{\epsilon+E} + \frac{1}{4}\,\frac{1}{\epsilon'+E}\right.$$

E is the energy of collision in the center of mass system and hence is one-half the bombarding energy if the protons may be considered at rest. The value of ϵ' needed to fit experiments is of the order of 0.10 Mev but the sign of ϵ' is not determined because $\sin^2\delta$ is determined from the square of the logarithmic derivative at a.

By taking into account the effect of the finite kinetic energy inside the distance a the expression for σ becomes, in next approximation

$$\sigma = \frac{4\pi\hbar^2}{M}\left\{\frac{3}{4}\,\frac{1+(\sqrt{M\epsilon}/\hbar)a}{\epsilon+E} + \frac{1}{4}\,\frac{1+(\sqrt{M\epsilon'}/\hbar)\,a}{\epsilon'+E}\right\}$$

Although the spherically symmetrical solution for the deuteron must be incomplete because of the existence of the quadrupole moment it represents the gross properties quite well. Before considering refinements of the theory, therefore, it is worthwhile to study the implied dependence of the force between neutron and proton upon their relative spin orientation. With parallel spins there is a binding energy, ϵ = 2.16 Mev and with opposite spins (singlet state) the binding energy is very close to zero. Thus the relation between V_0 and a^2 obtained in an approximate way above applies very closely for the singlet state. For the triplet the exact relation determining $V = V_1$ is, in the units used above,

$$V_1 = \frac{25.3}{a^2} + \frac{13.2}{a} + 2.4$$

Thus for <u>a</u> = 1 the potential well is almost twice as deep for the triplet state as for the singlet. This indicates that the spins of the nuclear particles play a dominating role in the attraction between neutron and proton.

LA 24 (27)

December 28, 1943

LECTURE SERIES ON NUCLEAR PHYSICS

Fourth Series: Two Body Problem Lecturer: C.L. Critchfield

LECTURE XXVII: I) SPIN-SPIN FORCES IN THE DEUTERON

II) PROTON-PROTON SCATTERING

I.

The theory that has been developed for the deuteron is generally satisfactory except for describing the electrical quadrupole moment. In order to account for the quadrupole moment Schwinger used a spin-dependent interaction similar to that between magnetic dipoles. The wave equation then becomes

$$\frac{\hbar^2}{M} \nabla^2 \psi(r, \theta, \varphi, S_N, S_p) + \left\{ E - V \left[\frac{\gamma\, 3(\vec{\sigma}_N, \vec{r})(\vec{\sigma}_p, \vec{r}) - r^2(\sigma_N, \sigma_p)}{r^2} \right] \right\} \psi = 0$$

where $\vec{\sigma}_N$ is the neutron spin operator (Pauli matrix), $\vec{\sigma}_p$ the proton spin operator and γ determines the amount of the spin-spin potential. Since the additional force introduces components of $\vec{r}$ in a non-spherical combination it is in general impossible to assume spherical symmetry for the solution. For the singlet state of the deuteron, however, the dipole-dipole interaction cannot affect the motion of the particles and its average value is zero so the same equation as before is obtained and the same relation between V_o and a^2 holds. For the triplet state the interaction does not vanish but rather tends to change the spin directions of the particles and hence also their orbital motion. The interaction is invariant to reflection in the center of symmetry so the orbital state into which the deuteron might go is limited to even parity and the only possibility is a D state. Symmetry of the operator in σ_N, and σ_p shows that the spins of the neutron and proton remain parallel and the solution will be a mixture of $^3S_1$ and $^3D_1$ states. The amount of $^3D_1$ that is mixed with the $^3S_1$ to obtain the lowest energy is determined by the size of γ which is, in turn, chosen to give the correct value of the quadrupole moment, which is 2.73×10^{-27} cm^2.

The observed quadrupole moment is positive and that means that the charge distribution is "cigar shaped". Mathematically the quadrupole moment is defined as the average of $(1/4)(3z^2 - r^2)$ over the charge distribution obtained from the solution of the wave equation. Qualitatively it is readily demonstrated that when two parallel dipoles interact the energy will be lowest when they lie on an axis and that such a configuration is brought about by interference between D waves and S waves. The part of the D wave that varies as $3 \cos^2 \theta - 1$ will interfere with the S wave because the spins are the same for the two waves. Along the axis, therefore, amplitudes add whereas the parts of the D wave that are large near $\theta = (\pi/2)$ $(m_L > 0)$ do not have identical spin magnetic quantum numbers and intensities add. The result is, with one phase of D wave, a prolate charge distribution about the spin axis and with the other phase an oblate charge distribution. The sign of γ is thus chosen to give the prolate distribution and the value of γ appropriate to a range of force 2.80×10^{-13} cm $(a = 1)$ is 0.775. The corresponding depth of potential well is $V_0 = 27\ mc^2$ which is close to that for the depth of the singlet well $V_0 = 25\ mc^2$. In fact Schwinger points out that if the range of forces had been taken 2.7×10^{-13} cm the two potentials would have been the same and the entire dependence upon spin of the neutron and proton is imparted by the dipole-dipole interaction. It may be noted here that if this dipole interaction had been due to the magnetic moments it would be of the opposite sign and its expected value much smaller than that found above.

The amount of ^{3}D-wave in the lowest state is found to be only 4 percent. The characteristic influence of the dipole forces on scattering, capture and photoelectric disintegration are correspondingly small, amounting in general to 2 percent corrections. Formulas obtained in the simple S wave theory are therefore adequate for most calculations and fail only in the true interpretation of the forces and in predicting the quadrupole moment. The remarkable effect of this evidently very small admixture of orbital motion is that essentially the same depth of potential well obtains for both triplet and singlet state. In other words, the difference in depth found necessary in the spherical states is just made up by the strength of the spin coupling but the ^3D state to which the ^3S is coupled is very hard to excite.

A reasonable theory and method of calculation is then available for the deuteron provided that the energies are not too high. At high energy of collision the potential well can no longer be considered small compared with a wave length and the shape of the potential curve will matter. This energy is roughly equal to the depths of the potential wells, say 10 Mev, in the center of mass system. Actually, the neutron-proton cross section has been observed at 24 Mev (12 Mev in the center of mass frame) by Sherr and the results are fairly well accounted for by the simple theory including P- waves. Above this energy the effects of the shape of the well and of the response to collisions with higher angular momentum should be in convincing evidence.

II.

The range of forces between neutron and proton has been taken to be 2.8×10^{-13} cm (a 1). There are two methods of establishing this value for the range. The first to be considered is the scattering of protons by protons and detection of deviations from the Rutherford law. Deduction of the range of forces in the deuteron from these results assumes that the ranges are the same. The second method involves making estimates of the masses of H^3, He^3 and He^4 from the theory of the deuteron.

As in the case of neutron-proton forces the experimental energies available are capable of finding only the nuclear forces exerted between two protons when they collide in an S-state. The interaction may therefore again be represented by a "square well" but calculation of the influence of such a "well" on the scattered distribution is somewhat more complicated for two charged particles There are three reasons for the complication: The first is that the scattering effect of the "well" is superposed on that of the electric fields so that interference terms appear in addition to those due to nuclear forces alone; the second reason is that the Coulomb forces are long range and modify the incident wave even at infinite distance so that the simple exponentials are not solutions at any distance from the point of collision. In addition, the collision of two protons if influenced by the statistics of the particles and a second kind of interference is introduced. The Fermi-Dirac statistics admit collision in a ^1S state and exclude the ^3S so we are here concerned only with the nuclear forces between protons of opposite spin.

The wave equation of the reduced system in the Coulomb field is $\frac{\hbar^2}{M}\nabla^2\psi(r,\theta,\varphi)+\left[\frac{Mv^2}{4}-\frac{e^2}{r}\right]\psi(r,\theta,\varphi)=0$

where v is the relative velocity at infinite separation. Let $k \equiv Mv/2\hbar$ and $\eta \equiv e^2/\hbar v$ and let $u(r)/r$ be the spherically symmetrical part of the solution. Then the equation for $u(r)$ becomes

$$\frac{d^2u}{dr^2}+\left(k^2 - \frac{2k\eta}{r}\right)u = 0.$$

If we replace u by ye^{ikr} and neglect y_{rr} compared with $2iky_r$ we get

$$iy_r - (\eta/r)y = 0$$

or

$$y = \text{const } x e^{-i\eta \ln kr}$$

and the asymptotic form of u is not simply e^{ikr} but

$$u \sim e^{ikr - i\eta \ln kr}$$

The same form is obtained, of course, for any angular momentum. In a similar way the asymptotic form of the incident beam can be shown to be

$$\psi_i \sim e^{ikz + i\eta \ln k(r-z)}$$

Following the example of the neutron-proton scattering we seek a solution in the form of an incident beam plus pure scattered wave. The asymptotic form must then be:

$$\psi(r,\theta) \sim e^{ikz + i\eta \ln k(r-z)} + f(\theta)\frac{e^{ikr - i\eta \ln kr}}{r}$$

where θ is the angle in the center of the gravity system. Deflections by θ in that system are measured as deflections of $(1/2)\theta$ in the laboratory. Derivation of $f(\theta)$ is a long mathematical process and will not be presented here. Suffice it to say that the result of the wave mechanical calculation for dissimilar particles is identical with the classical calculation of Rutherford. This means that

$$\sigma_R(\theta) = \left|f(\theta)\right|^2 = \left(\frac{e^2}{Mv^2}\right)^2 \frac{1}{\sin^4 (1/2)\theta}$$

Furthermore, the phase of $f(\theta)$ can be determined as a function of an angle by considering that the wave scattered directly back will have the same asymptotic exponential form as ψ_1 except for a change in sign. This is the case if

$$f(\theta) = -\frac{e^2}{Mv^2}\csc^2\tfrac{1}{2}\theta\, e^{-i\eta\ln(1-\cos\theta)+2i\beta}$$

The sign and the phase, β, are not given by the considerations but β is a function of velocity only. The complete theory shows

$$2i\beta = \ln\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}$$

The spherical portion of the wave in a Coulomb field has the asymptotic form

$$u = e^{i\beta}\,\frac{\sin(kr-\eta\ln 2kr+\beta)}{kr}$$

The effect of the forces between protons will be to introduce a constant phase shift K_o in such a way as to leave the incoming part of u unchanged. The form of the wave that has been influenced by the nuclear forces is therefore

$$u_f = e^{i\beta+iK_o}\,\frac{\sin(kr-\eta\ln 2kr+\beta+K_o)}{kr}$$

The asymptotic form of the perturbed wave then becomes

$$\psi(r,\theta) = e^{ikz\,+\,i\eta\ln k(r-z)}$$
$$-\frac{e^{ikr\,-\,i\eta\ln kr}}{r}\left\{\frac{e^2}{Mv^2\sin^2(1/2)\theta}\,e^{-i\eta\ln(1-\cos\theta)+2i\beta}\right.$$
$$\left.-\frac{1}{2iK}\left(e^{2iK_o}-1\right)e^{2i\beta-i\eta\ln 2}\right.$$

If the particles were not identical the cross section would be given by the absolute square of the curly brackets. In classical theory, the cross section for identical particles is the sum of the cross sections $\sigma(\theta) + \sigma(\pi + \theta)$. But in quantum theory there is an additional consideration to be made. If one proton is at r_a and the other at r_b the wave function describing the state is

$$\psi_1(r_a)\ \psi_2(r_b) + \psi_1(r_b)\ \psi_2(r_a)$$

in the singlet spin state and $\psi_1(r_a)\ \psi_2(r_b) - \psi_1(r_b)\ \psi_2(r_a)$ in the triplet. Therefore (since $r_a(\theta) = r_b(\pi+\theta)$) the scattering at $\pi-\theta$ which is thesame as at $\pi+\theta$ interferes constructively with that at θ in the singlet collisions, i.e. 1/4 of the time, and interferes destructively in triplet collisions (3/4). If we call the quantity in curly brackets $f_v(\theta)$ the cross section for scattering in unit solid angle at θ then becomes

$$\sigma = \frac{1}{4}\left| f_v(\theta) + f_v(\pi - \theta) \right|^2 + \frac{3}{4}\left| f_v(\theta) - f_v(\pi - \theta) \right|^2$$

This effect was first discussed by Mott.

$$\sigma(\theta) = \left(\frac{e^2}{Mv^2}\right)^2 \left\{ \frac{1}{\sin^4 (1/2)\theta} + \frac{1}{\cos^4 (1/2)\theta} - \frac{\cos(\eta \ln \tan^2 (1/2)\theta}{\sin^2(1/2)\theta \cos^2(1/2)\theta} \right\}$$

$$+ \frac{\sin^2 K_o}{k^2} - \frac{\sin K_o}{k}\left[\frac{\cos\ (\eta \ln \sin^2\ (1/2)\theta + K_o)}{\sin^2\ (1/2)\ \theta} \right.$$

$$\left. + \frac{\cos(\eta \ln \cos^2 (1/2)\theta + K_o)}{\cos^2 (1/2)\theta} \right] \left(\frac{e^2}{Mv^2}\right)$$

The first term is the pure Coulomb scattering, the second the pure potential scattering and the last the interference term. To get the cross-section for scattering between θ and $\theta + d\theta$ multiply by $2\pi \sin \theta d\theta$; and to obtain the cross section in the laboratory system replace every θ by 2θ. The largest effect of the potential well comes at 90° in the center of mass system (45° in the laboratory) and at this angle

$$\sigma\left(\frac{\pi}{2}\right) = \left(\frac{e^2}{Mv^2}\right)^2 \left\{4 + 4 \frac{\sin^2 K_o}{\eta^2} - 8 \frac{\sin K_o}{\eta} \cos K_o\right\}$$

$$\frac{\sigma\left(\frac{\pi}{2}\right)}{\sigma_c\left(\frac{\pi}{2}\right)} = 1 - \frac{\sin 2 K_o}{\eta} + \frac{\sin^2 K_o}{\eta^2}$$

At a million volts collision energy η is about 1/6 and it becomes smaller as the energy increases. The effect of a small phase shift due to nuclear forces is therefore greatly amplified at these energies. Further, due to interference, the sign of K_o is determined.

Experiments on proton-proton scattering have been carried out with good precision by Tuve, Hafstad, Heydenburg and Herb, Kerst, Parkinson and Plain. The latter have carried the collision energy up to 2.4 Mev where the ratio of cross sections is 43 and K_o is 48°. At 1 Mev $K_o = 33^\circ$. Both experimental groups accelerated protons by electrostatic generators and scattered them in hydrogen gas making counts at several angles to the beam.

Comparison of the results obtained with those that are expected from a square well can now be made. For this purpose, however, it is necessary, in general, to know both the bounded and unbounded solutions to the wave equation in the Coulomb field. These solutions are called F and G respectively. The phase shift, K_o, has been defined to relate to the asymptotic forms of F_o and G_o for S-waves which are taken to be $\pi/2$ out of phase at large r. At the boundary of the well the combination F_o cos $K_o + G_o$ sin K_o must fit onto the wave function, F_i, that applies inside the well

$$\frac{F'_i}{F_i} = \frac{F'_o + G'_o \tan K_o}{F_o + G_o \tan K_o}$$

and this determines K_o. If we assume that the Coulomb field has not affected F_o and G_o greatly we get the usual relation

$$r_o \frac{\sqrt{M(E+V)}}{\hbar} \cot \frac{\sqrt{M(E+V)}}{\hbar} r_o = r_o \frac{\sqrt{ME}}{\hbar} \cot\left[\frac{\sqrt{ME}}{\hbar} r_o + K_o\right]$$

With $r_o = 2.8 \times 10^{-13}$, $K_o = 48^\circ$, $E = (1/2)(2.4$ Mev) we find V_o = 9.0 Mev. Breit and others have shown that about 0.8 Mev should be added as an average effect of the Coulomb repulsion inside the well The more exact calculation gives 11.3 Mev as the depth of the well and also shows that good agreement with results at all energies is obtained. The rough method used would be applicable only at energies very high compared with the Coulomb repulsion at $r_o = e^2/mc^2$.

Using the exact wave functions a search for the best value of r_o has been made by Breit et al and the decision reached that $r_o = e^2/mc^2$ is the most acceptable ($a = 1$). A value of $a = .75$ gives a noticeably inferior fit as also does $a = 1.25$. The depth of well pertaining to $a = 1$ is 11.3 Mev in good agreement with the most careful estimates of the depth of the singlet deuteron well of the same width, 11.6 Mev.

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LECTURE SERIES ON NUCLEAR PHYSICS

Fourth Series: Two Body Problem   Lecturer: C. L. Critchfield

LECTURE XXVIII: THREE BODY PROBLEMS

A second method of determining the range of nuclear forces is obtained in estimating the binding energy of the nuclei containing three particles, H^3 and He^3. The binding energies of these nuclei are approximately equal, the difference being caused by the electro static repulsion of the two protons in He^3. Since an attractive force between two protons has been demonstrated in scattering experiments, it follows from the equality of binding in H^3 and He^3 that the same attractive force exists between neutrons. Attractive forces between all particles in these nuclei and in He^4 are also found necessary in getting agreement between observation and the estimates.

The method of calculating binding energies that we shall apply is to reduce the three-body problem to an "equivalent two body problem" by certain assumptions and by averaging over the third particle. The method was first used in this connection by Feenberg. Consider the nucleus H^3 which has a binding energy 16.3 mc^2. Assume that the wave function that describes any two particles in H^3 is the same as for the deuteron. In order to express the deuteron wave function as a single algebraic function the form of the two particle function will be assumed to be

$$f(r) = e^{-(1/2)(\nu r^2)}$$

where $(2/\nu)^{1/2}$ is essentially the radius of the wave of the deuteron. Further assume that the wave function for all three particles is a symmetric product of three such deuteron functions in the three separations r_{12}, r_{13} and r_{23}:

$$\psi = f(r_{12})f(r_{13})f(r_{23})$$

The potential energy of each particle will depend upon the position of each of the other two; but an effective two body potential can be obtained by averaging over the density function of one of the particles. The potential energy may be written (V is the deuteron potential):

$$V = \frac{\int \psi \left[V(r_{12}) + V(r_{13}) + V(r_{23})\right] \psi \, d\tau_{12} d\tau_3}{\int \psi^2 d\tau_{12} d\tau_3} = 3 \frac{\int \psi V_{12} \psi \, d\tau_{12} d\tau_3}{\int \psi^2 \, d\tau_{12} d\tau_3}$$

Carrying out the integral over $d\tau_3$ (the volume element of particle 3) we get

$$\bar{V} = 3 \frac{\int e^{-(3/2)(\nu r_{12}^2)} V(r_{12}) \, d\tau_{12}}{\int e^{-(3/2)(\nu r_{12}^2)} d\tau_{12}}$$

Thus, if we set $r^2 = (3/2)(r_{12}^2)$, the potential between any two particles is the same as in the deuteron except that the range of forces is $\sqrt{3/2}$ larger. The total potential is three times that between any two particles. In a similar calculation the kinetic energy may be averaged and the result is that the total kinetic energy is three times that in the deuteron.

We shall apply the results obtained above to determining the equivalent solution with a "square" potential well. Let $_T$ be the binding energy of H^3; the wave equation for the two body problem "equivalent" to H^3 is then

$$3 \frac{\hbar^2}{M} \frac{d^2u}{dr^2} - \left[3V\left(\sqrt{\tfrac{3}{2}}\ a\right) + \epsilon_T \right] u = 0$$

The function $V(\sqrt{\frac{3}{2}}\ a)$ is equal to a constant, $-V_T$, for $r < \sqrt{\frac{3}{2}}\ a$ and vanishes for $r > \sqrt{\frac{3}{2}}\ a$. In the three particle nuclei V_T is a mixture of singlet and triplet potentials. The two neutrons is H^3 are certainly in a singlet state and if the spin of the proton is parallel to one of the neutron spins it forms the triplet with that neutron and half triplet and half singlet with the other. The expected relation between V_T and a can be determined from the deuteron calculations and is a simple average of singlet and triplet relations

$$V_T = \frac{25.3}{a^2} + \frac{6.6}{a} + 1.2 \qquad (H^2)$$

This average is based on the assumption that the effect of the D wave in H^3 is the same between pairs of particles in the triplet state as in the deuteron.

The other relation between V_T and a may be obtained from the equivalent two body equation for H^3. That equation may be made formally the same as for the triplet deuteron by substituting

$$r = \lambda r'$$

and choosing λ^2 such that

$$\lambda^2 \frac{\epsilon_T}{3} = \epsilon_D$$

where ϵ_D is the binding energy of the triplet deuteron. Then the relation between $\lambda^2 V_T$ and $\sqrt{\frac{3}{2}}\ \frac{a}{\lambda}$ is the same as between V_o and a for the triplet deuteron, i.e.

$$\lambda^2 V_T = \frac{25\,\lambda^2}{\frac{3}{2}\,a^2} + \frac{13.2\lambda}{\sqrt{\frac{3}{2}}\,a} + 2.4$$

The appropriate value of λ^2 is 0.78 and

$$V_T = \frac{16.9}{a^2} + \frac{12.2}{a} \quad 3.2 \qquad (H^3)$$

This, combined with the relation obtained from the deuteron (H^2) has the solution

$$a = 1.08$$

for equal depths of well, V_T. The value for the range of forces thus obtained is higher than that indicated by proton-proton scattering but only by 8 percent. The method is very rough, of course, and the result obtained is satisfactory. If a neutron-neutron force had not been postulated the agreement would be unacceptable.

There is a difference in binding between He^3 and H^3 of about .76 Mev that should be due to the Coulomb repulsion of the protons. We may estimate the expected repulsion as follows:

$$\Delta E = \frac{\int_0^\infty e^{-(3/2)(\nu r^2)}(e^2/r)(r^2dr)}{\int_0^\infty e^{-(3/2)(\nu r^2)}\,r^2dr} = e^2\left(\frac{6\nu}{\pi}\right)^{1/2}$$

Now $\nu/2$ represents the reciprocal square of the extent of the wave function and may be taken approximately equal to $M\epsilon/\hbar^2 = 0.43$ in units of m^2c^4/e^4. This gives 0.65 Mev for the calculated Coulomb energy in satisfactory agreement with the experimental value considering the approximations made.

Similar three and four body calculations have been made by the variational method with substantially the same result for a and a correct accounting of the binding energies of H^3, He^3 and He^4.

There is another type of three-body problem of fundamental interest to the theory of the deuteron. This is the scattering of neutrons by hydrogen molecules. The theory has been worked out by Schwinger and Teller and has shown that the scattering cross section of the molecule is extremely sensitive to whether the singlet state of the deuteron is real or "virtual". Experiments on scattering of liquid-air neutrons have been interpreted by the theory to prove conclusively that the singlet state is virtual, i.e., there is no bound singlet state.

The great decisiveness of the experiment comes about because of an accidental near cancellation of terms in the calculation. Details of the calculations will not be given here but the principal feature of them is readily described. The calculated cross section for neutron-proton scattering due to the triplet well alone is 3.50×10^{-24} cm^2 at low energy whereas the observed value is 13×10^{-24} cm^2. From Wigner's hypothesis the singlet scattering must have a cross section 41.5×10^{24} cm^2. If σ_1 is the triplet and σ_0 the singlet cross section;

$$\sigma_0 \quad 11.8\ \sigma_1$$

Scattering from the hydrogen molecule will show interference between spherical waves coming from the two nuclei if the wave length of the neutron is long compared with the internuclear separation. At room temperature the wave length is of the same order of magnitude as the separation. Calculation of the cross section for neutrons of this energy and below will then involve the square of a quantity representing the sum of effects from the two protons. If both protons scatter in, say, the triplet state the phases have the same sign and interfere constructively but if one scatters in the triplet and one in the singlet the waves will reinforce only if the singlet level is real. If the singlet level is virtual the phase shift will be of opposite sign to that from the triplet and the waves will interfere destructively.

A somewhat more quantitative analysis may be made by introducing the phase shifts δ_1 and δ_0 for very slow neutrons leading to the triplet and singlet cross sections respectively.

$$\sigma_1 = \frac{4\pi}{k^2} \sin^2 \delta_1 \qquad \sigma_0 = \frac{4\pi}{k^2} \sin^2 \delta_0 \qquad \sin\delta_0 = \pm 3.43 \sin\delta_1$$

The equations for σ_1 and σ_0 can be combined into one equation by use of the scalar product of neutron and proton spin operators $\vec{\sigma}_N \cdot \vec{\sigma}_p$. This product has the value +1 for a triplet state and -3 for a singlet state.

$$\sigma = \frac{\pi}{4} \frac{1}{k}^2 \left[3 \sin\delta_1 + \sin\delta_0 + (\sin\delta_1 - \sin\delta_0)\, \vec{\sigma}_N \cdot \vec{\sigma}_P\right]^2$$

The cross section for scattering by two protons, 1 and 2 forming a molecule is proportional to the square of the sum of the expressions in the bracket for each proton provided that the protons are in the same state of symmetry after scattering as before and provided the wave length of the neutron is very long. The symmetry of the proton state remains unchanged unless there is a conversion from orthohydrogen to parahydrogen or vice versa. Thus the cross section for ortho-ortho or para-para scattering is proportional to

$$\sigma \sim \left[3 \sin \delta_1 + \sin \delta_0 + (\sin \delta_1 - \sin \delta_0)\ \vec{\sigma}_N \cdot \frac{\vec{\sigma}_1 + \vec{\sigma}_2}{2}\right]^2$$

the square should be averaged over all neutron spins.

$$\sigma \sim (3 \sin \delta_1 + \sin \delta_0)^2 + (\sin \delta_1 - \sin \delta_0)\ \frac{3 + \vec{\sigma}_1 \cdot \vec{\sigma}_2}{2}$$

For para-para scattering

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = -3$$

and

$$\sigma_{pp} \sim (3 \sin \delta_1 + \sin \delta_0)^2$$

which amounts to 41 if the singlet state is stable, i.e., δ_0 and δ_1 have the same sign, and only to 0.18 if the singlet is unstable. There is thus a factor of over 200 in cross section depending upon the stability of the singlet state and the large size of the factor is due to the fact that $\sin \delta_0$ is so nearly equal to -3 times $\sin \delta_1$. For ortho-ortho scattering $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = +1$

$$\sigma_{oo} \sim (3 \sin \delta_1 + \sin \delta_0)^2 + 2(\sin \delta_1 - \sin \delta_0)^2$$

The right side is 53 for the stable singlet and 20 for an unstable singlet.

The experimental procedure is to compare the scattering of para-hydrogen with that of orthohydrogen (or rather the usual mixture) for liquid air neutrons. The interpretation is subject to the further complication, however, that the neutrons may convert one kind of hydrogen into the other. In this case the wave function of the final molecule has one sign if the spin of one proton is changed and the opposite sign if the spin of the other is changed. The interference is then destructive and the cross section proportional to

$$\sigma_{conv} \sim (\sin \delta_1 - \sin \delta_0)^2$$

Of course the conversion is accompanied by a change in rotational state because of the statistics. Schwinger and Teller have shown that σ_{conv} is of the same order of magnitude as σ_{oo}, so that only when the parahydrogen is mostly in the ground state, J = 0, and when the neutrons have sufficiently low energy will its cross section be so extremely small for the case of an unstable singlet deuteron. The energy required to raise a para molecule to ortho state is .023 electron volts, hence liquid air neutrons (.012 ev) will be unable to convert para to ortho hydrogen. The result of the experiment is that the scattering cross section of ortho-H_2 is much larger than that of para hydrogen. This proves conclusively that the singlet deuteron is not stable.

Exactly the same kind of considerations have been made by Schwinger for the possibility that the neutron has spin $(3/2)\hbar$ instead of $(1/2)\hbar$. The level of the deuteron that lies near zero energy would then be a quintet. In this case however, the fortuitous cancellation of terms does not occur and all cross sections in molecular hydrogen are of the same order of magnitude. Thus the experiments prove not only that the singlet state of the deuteron is unstable but also that the neutron spin is $(1/2)\hbar$.

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LECTURE SERIES ON NUCLEAR PHYSICS

Fourth Series: Two Body Problem Lecturer: C.L. Critchfield

LECTURE XXIX & XXX: INTERACTION OF NEUTRON AND PROTON WITH FIELDS

Neutrons and protons interact with the fields that have been established by classical physics, namely, the gravitational fields and the electromagnetic fields. Gravitational phenomena are of no consequence in the description of nuclear processes. The average gravitational attraction in the deuteron amounts to only 4×10^{-30} electron volt. Electromagnetic phenomena, on the other hand, are very important. Radiative capture, emission of gamma rays following transmutation and photoelectric dissociation are evidences of the response of nuclei to electric and magnetic fields. The neutron is affected only through its magnetic moment and the proton through both its electric charge and its magnetic moment.

In addition to these well established fields there are two, still incomplete, theories of fields that influence the nuclear particles. One of these is based upon an interaction between neutron or proton with the field of electron and neutrino waves. The purpose of this theory is to systematize observations on the beta-activity of nuclei. The other theory postulates an interaction between the nuclear particle and the field of mesons, particles discovered in cosmic radiation having a mass about 200 times that of the electron. The purpose of this theory is to account for nuclear forces and for the magnetic moments of neutron and proton. In certain forms of the latter theory the nature of beta activity is circumscribed to some extent. Some fundamental examples of the applications that are made of the three important field theories will be considered.

Electromagnetic Fields

It has been pointed out in the lectures on the neutron that the best determination of the mass of the neutron is a result of the photodisintegration of the deuteron. Also, accurate experimentation on photodisintegration, especially on the angular distribution of the separating particles, will be valuable in understanding details of the structure of the light nuclei. The theory of the interaction between the deuteron and electromagnetic radiation is consequently of considerable importance and a brief sketch of its application will be given.

The effect of an electromagnetic field on nuclear system can be treated as a small perturbation of the energy of the system. In general, the fields act on both the electric and magnetic moments of the nucleus. We shall consider the electric interaction with the deuteron first.

The perturbation energy is equal to the product of the electric field strength E and the electric dipole moment $e(\underline{r}/2)$ in the deuteron. Of course, the average value of the latter vanishes but the electric dipole is operative in causing transitions from the, essentially spherically symmetric, ground state of the deuteron, to P-states which conserve the energy given by an incident photon and representing the ejected particles. The P-states are unbound, and in fact very little affected by the nuclear forces because of the vanishing intensity of the wave at small separations of the particles. If we normalize the P-wave in a sphere of radius R which is large compared with the wave length of the ejected particles the number of processes per second is given by the well known result of perturbation theory:

$$w = (2\pi/\hbar)\left|\int \psi_s(e\underline{r}/2 \cdot \underline{E})\,\psi_p\, d\tau\right|^2 P_p$$

where P_p is the number of P-states per unit energy at the energy of the ejected particles, ψ_s is the wave function of the ground state of the deuteron and ψ_p the P-wave. For simplicity we shall assume ψ_s in the form

$$(1/r)e^{-(\sqrt{M\epsilon}/\hbar)\,r}$$

normalized in a very large sphere.

$$\psi_s = (\sqrt{M\epsilon}/2\pi\hbar)^{\frac{1}{2}}\left[(1/r)e^{-(\sqrt{M\epsilon}/\hbar)\,r}\right]$$

In the case of the P-wave the radius of the sphere appears explicitly

$$\psi = \frac{1}{\sqrt{2\pi R}}\quad\frac{1}{r}\left(\frac{\sin kr}{kr} - \cos kr\right)\quad(\sqrt{3}\cos\theta)$$

Only the P-wave that represents the direction of polarization will contribute to the integral in w.

The density of states per unit energy is obtained in the following way. Quantization in a sphere of radius R allows only those values of k for which, essentially, cos kR = 0. Since $k = P/\hbar$ there are thus $R/\pi\hbar$ states per unit momentum and

$$P_p = (R/\pi\hbar)\,(dp/dE) = R/\pi\hbar v$$

per unit energy; v is the relative velocity of the separating particles.

It is possible to divide w into the factors

$$w = 2\pi/\hbar./M_E/^2.E_p{}^2 \cdot P_p$$

so that E^2 represents the density of photons of polarization p at the nucleus and M_E is the nuclear "matrix element". This can be done since the wave length of the radiation is very large compared with nuclear dimension and thus E^2 is sensibly a constant factor in the integral. The incident current expressed as number of photons of both polarizations per second per square centimeter is

$$I = c(E^2 + H^2/8\pi h\nu) = cE^2/4\pi h\nu$$

and the cross section for photo electric disintegration (only one polarization is effective)

$$\sigma_E = \tfrac{1}{2}w/I = (8\pi^3\nu/c)\left|M_E\right|^2 P_p$$

If we had normalized the neutron-proton P-wave functions to unit energy the factor P_p would have been unity.

Calculation of M_E is straightforward. Consider the dipole moment in the direction, $\theta = 0$,

$$M_E = \int \psi_s(er/2 \cos\theta)\psi_p d\tau$$

$$M_E = (2\xi k^2)/\left[(M\epsilon/\hbar^2) + k^2\right]^2 \quad \left[(\sqrt{M\epsilon})/(3\hbar R)\right]^{\frac{1}{2}}$$

and with $\qquad v = 2p/M = 2k\hbar/M$

$$\sigma_E = (16\pi^2/3)\ (\mu/c)(\epsilon^2 M/\hbar^2) \quad \frac{(\sqrt{M\epsilon}/\hbar)k^3}{\left[(M\epsilon/\hbar^2) + k^2\right]^4}$$

Now $h\nu = E + \epsilon = \left[(\hbar^2 k^2)/M\right] + \epsilon$, where E is the kinetic energy of neutron and proton and σ, expressed as a function of E and ϵ becomes

$$\sigma_E = (8\pi/3)(e^2/\hbar c)(\hbar^2/M\epsilon)\left(\frac{\sqrt{E/\epsilon}}{1 + E/\epsilon}\right)^3$$

For the γ-rays of Th C', used by Chadwick and Goldhaber, the calculated cross section is 6.7×10^{-28} cm^2

To the cross section σ_E must be added the cross section due to effect of the magnetic vector, σ_M. The perturbing energy is then $\mu \cdot \underline{H}$, and the matrix element, M_m, contains not only an integral over space coordinates but a sum over the spins. The transition is from the ^3S ground state to a ^1S state of positive energy. Since there is a virtual level of the deuteron in the positive spectrum, the ^1S wave function is very large at low energy and this fact makes up for the relative weakness of the magnetic moments. Assume the proton and neutron magnetic moments to be independent, then

$$M_m = ({}^3S \mid \mu_p \sigma_p + \mu_n \sigma_n \mid {}^1S) \int \psi({}^3S)\, \psi({}^1S)\, d\tau$$

The spin function for the singlet may be written symbolically

$$^1S = (1/\sqrt{2})(p\uparrow n\downarrow - p\downarrow n\uparrow)$$

and $\sigma_p\, {}^1S = -\sigma_n\, {}^1S$. Furthermore, the triplet spin state is required to cancel the angular momentum of the light quantum and therefore only one of three possible triplet states contributes. If, for example, this one state is

$$^3S = (1/\sqrt{2})(p\uparrow n\downarrow + p\downarrow n\uparrow)$$

the value of $({}^3S \mid \mu_p \sigma_p + \mu_n \sigma_n \mid {}^1S)$ is seen at once to be $\mu_p - \mu_n$.

In the space-integral for M_m we use $\psi({}^3S) = \psi_s$ as above, but

$$\psi({}^1S) = \frac{\sin(kr + \delta_0)}{r\sqrt{2\pi R}}$$

where δ_0 is the phase shift caused by the potential well

$$\cot \delta_0 = \sqrt{M\epsilon^1}/\hbar k$$

$$M_m = 2(\mu_p - \mu_n)(\sqrt{M\epsilon}/\hbar R)^{\frac{1}{2}} \frac{\sqrt{M\epsilon}/\hbar + \sqrt{M\epsilon^1}/\hbar}{+(M\epsilon^1/\hbar^2 + k^2)^{\frac{1}{2}} \; (M\epsilon/\hbar^2 + k^2)^{\frac{1}{2}}}$$

$$\frac{M_m}{M_E} = \frac{\mu_p - \mu_n}{2ek} \sqrt{3} \; \left(\frac{\sqrt{\epsilon} + \sqrt{\epsilon'}}{(E + \epsilon')}\right) \frac{1}{2} \quad (E + \epsilon) \frac{M}{\hbar^2}$$

Now, only one of the three possible orientations of a deuteron is capable of absorbing the light quantum. The ratio of cross sections is then

$$\sigma_M/\sigma_E = 1/3(M_m/M_E)^2 = (\mu_p - \mu_n)^2/4 \left[(\sqrt{\epsilon} + \sqrt{\epsilon'})^2 \ (E+\epsilon)^2\right]\left[(E+\epsilon^1)Mc^2 \ E\right]$$

where μ_p and μ_n are now measured in nuclear magnetons, $\mu_p - \mu_n = 4.72$. For ThC^1 γ-rays E = 0.46 MV. and ϵ = 0.11 MV., $\sigma_M/\sigma_E = .51$. For this energy γ-ray (2.62 MV). Therefore, the cross sections are comparable in magnitude. At higher energy the electric effect should predominate and at lower energy the magnetic effect is stronger. The angular distribution from the two effects differs also. A spherical wave is created by the magnetic interaction but the electric interaction sends the neutron and proton out in P-waves that have a nodal plane parallel to the direction of radiation. The angular distribution is therefore

$$\sigma(\sigma) = 3/2 \sin^2\sigma \, \sigma_E + \sigma_M$$

and the ratio of neutron or proton intensities at $\sigma = 0$ and $\sigma = 90°$ to the γ ray beam is

$$\sigma(0)/\sigma(90°) = \sigma_M/(\sigma_M + 3/2\,\sigma_E) = .25$$

Having calculated the cross section σ_{AB} for a process in which the two particle system A is transformed into another, B, it is a simple matter to obtain the cross section σ_{BA} of the reverse process. Let g_A, g_B be the statistical weights of systems A and B respectively and λ_{AB}, λ_{BA} the wave lengths of the incident particles in processes AB and BA. It can be shown quite generally that

$$\sigma_{BA} / \sigma_{AB} = (g_A / g_B) \; (\lambda^2{}_{BA} / \lambda^2{}_{AB})$$

The reverse of photo disintegration is simple radiative capture of neutron by proton. System A is composed of one photon and the stable deuteron; the photon is capable of two polarizations and the deuteron of three, hence $g_A = 6$. System B contains one neutron and one proton each of two possible polarizations so $g_B = 4$. The cross section for capture is then

$$\sigma_c = 3/2(2\pi\nu/kc)^2 \quad (\sigma_E + \sigma_M)$$

The electric and magnetic effects have different dependencies on the energy of the neutron-proton system. At high energy the magnetic effect is negligible and approximately

$$\sigma_c \cong 3.7 \times 10^{-29} \sqrt{\epsilon/E} \text{ cm}^2 \qquad E \gg \epsilon$$

At very low energy, however, the magnetic effect predominated and

$$\sigma_c = 1.4 \times 10^{-29} \sqrt{\epsilon/E} \qquad E \ll \epsilon^1$$

For neutron at room temperature, 1/40 e.v., E = 1/80 e.v.

$$\sigma_c = .18 \times 10^{-24} \text{ cm}^2$$

FERMI'S THEORY OF BETA-RADIATION

When the neutron was discovered the way was opened for the development of a field theory of beta radiation. This was done by Fermi in analogy to electromagnetic theory. The fundamental assumptions are:

(1) Neutron and proton are two states of the same particle (nucleon).

(2) Energy, spin and statistics are conserved in beta radiation by the introduction of the Pauli neutrino.

(3) The rest mass of the neutrino is zero (or nearly

zero) and its spin is $\frac{1}{2}\hbar$.

(4) Neutron and proton interact with the combined fields of electrons and neutrinos in such a way that an electron and a neutrino are radiated when neutron changes to proton and a positron and a neutrino are radiated when proton changes to neutron.

(5) Electron and neutrino share the available energy in all possible proportions.

In analogy to the interaction between charged particles and light quanta, the number of beta processes per second is

$$w = (2\pi/\hbar)\, g^2 \left| M \right|^2 \psi_e^2 \, \psi_\nu^2 \rho_e \rho_\nu$$

where M is the matrix element calculated from waves of neutron and proton. The interaction constant g is analogous to e in electric theory. ψ_e and ψ_ν are the amplitudes of electron and neutrino waves at the nucleus, $\psi^2_e \psi_\nu^2$ takes the place of E^2 in the electromagnetic case. ρ_e and ρ_ν are the numbers of available electron and neutrino states per unit energy.

The strongest beta activities will be those emitting the electron and neutrino in S-waves since ψ^2_e and ψ_ν^2 will then be largest. For light nuclei these S-waves are not greatly affected by the Coulomb field if the electron has several million volts energy and we are justified in approximating ψ_e as

$$\psi_e = \frac{(\sin)\ (pr/\hbar)}{r\ \sqrt{2\pi R}}$$

where ψ_e is normalized in a sphere of radius R. Let the total energy liberated by the nucleus during radiation be W and let the energy of the electron be E. Then the wave function for the neutrino is

$$\psi_\nu = \frac{\sin (W-E)r/\hbar c}{r\sqrt{2\pi R}}$$

and, as in the case of the photoelectric disintegration of the deuteron

$$\rho_e = R/\pi\hbar v = RE/\pi\hbar c^2 p \qquad \rho_\nu = R/\pi\hbar c$$

at $r = 0$ $\psi_e = p/\hbar\sqrt{2\pi R}$ $\quad \psi_\nu = (W-E)/\hbar c\sqrt{2\pi R}$ and

$$w = (g^2 |M|^2/2\pi^3 c^5 \hbar^7)\ pE(W-E)^2$$

The shape of the electron spectrum is thus determined. Experimentally the number per unit energy (or unit momentum) is measured, divided by pE or its equivalent in the Coulomb field and the square root of the resulting number plotted against E. In the carefully done work with nuclei that have only one beta activity the plot is a good straight line, the intercept of which W. The theory is therefore substantiated by the experiments.

If the nuclear charge is larger than, say 20, or if the electron energy is particularly low the sine wave is not a good approximation for the electron wave. A more exact value of ψ_e^2 at $r = 0$ is

$$\psi_e^2 = (2\pi\eta/e^{2\pi\eta}-1)(p^2/\hbar^2)(1/\sqrt{2\pi R})$$

$$\eta = Ze^2/\hbar v$$

Z is plus the charge number of the residual nucleus if a positron is emitted and minus the charge number for electron emission.

The probability of emission of an electron with energy between E and E + dE is

$$P(E)\,dE = \frac{g^2}{2\pi^3\ c^5\ \hbar^7}\ |M|^2\ \frac{2\pi\eta}{e^{2\pi\eta}-1}\ PE\ (W-E)^2\ dE$$

The total probability of emission of electron and neutrino per second is given by the integral of $P(E)dE$ over electron energies from $E = mc^2$ to $E = W$. For $W \gg mc^2$ this integral is closely equal to

$$\lambda = \frac{g^2 \left|M\right|^2}{60\pi^3 c^6 \hbar^7} \quad \frac{2\pi Z/137}{e^{2\pi Z/137}-1} \, W^5$$

The decay constant, λ, is proportional to the fifth power of the total energy released by the transition. Measurements of W and the half lives of the light radioactive elements makes possible a determination of the "Fermi constant", g.

There is a class of positron emitters that is particularly well suited for determining g. This class comprises the light radioactive nuclei containing one more proton than neutron. The extra proton turns into a neutron with the emission of a positron and, since the nuclear forces are evidently the same between all pairs of particles, the wave function of the neutron in the final nucleus should be very nearly the same as the wave function of the initial proton. In this case $\left|M\right|^2 = 1$. A table of these nuclei, their half lives and energies released are given in Table I on the following page. In some instances two energies are given representing two possible final states. The life time is determined by the transition releasing the most energy. If the transition goes to the excited state the nucleus subsequently emits a gamma ray. An average value of $\lambda(mc^2/W)^5$ is 4.61×10^{-6}. If we measure g in units of $(e^3/Mc^2)^2$ we find

$$g = 46 \, (e^3/Mc^2)^2$$

TABLE I

NUCLEUS	HALF LIFE	MAXIMUM KINETIC ENERGY (MV)
C^{11}	21.0 m	1.15
N^{13}	9.93 m	.92, 1.20
O^{15}	126.0 s	1.7
F^{17}	70.0 s	2.1
Ne^{19}	20.3 s	2.20
Mg^{23}	11.6 s	2.82
Al^{25}	7.0 s	1.7, 3.0
Si^{27}	4.92 s	3.54
P^{29}	4.6 s	3.63
S^{31}	3.18 s	3.87
Cl^{33}	2.4 s	4.13
A^{35}	1.88 s	4.41
Sc^{41}	.87 s	4.94

We have assumed that electron and neutrino are emitted without orbital angular momentum. The maximum amount of spin that can be taken by the emitted particles is, therefore, $\hbar$. In order that the spin of the nucleus be changed it would be necessary that the coupling between neutron (or proton) and the field contain the spin operator, just as in the case of magnetic dipole capture of neutron by proton. Fermi's original theory did not contain such a spin operator and the nuclear spin could not change in "allowed" transition. In order to make a transition from one spin to another the neutrino or electron or both had to be emitted with orbital momentum. The amplitudes of waves of higher angular momentum are much smaller at the nuclear radius than the amplitude of the

S-wave and such transitions are termed "forbidden". The half-life of a forbidden transition is therefore much longer than that of an allowed transition of the same energy release.

Certain nuclei produce allowed transitions although it appears certain that the nuclear spin changes by $\hbar$. This is particularly true of He^6 which must have spin 0 but decays at an "allowed" rate to Li^6 with spin $\hbar$. In Fermi's theory such a transition would be forbidden. To account for transitions with change of spin ±1, Gamow and Teller postulated a spin dependent interaction with the electron-neutrino field.

The calculations made thus far apply only to allowed transitions in light nuclei. To extend them to heavy nuclei and to forbidden transitions more exact account must be taken of the wave functions for relativistic motion in a Coulomb field and there are several interesting features of the solution that are emphasized in beta-decay theory.

FIELD THEORIES OF NUCLEAR FORCES

The success of Fermi's theory of beta decay prompted an attempt to explain the forces between nuclear particles by the same interaction. Thus the electron-neutrino field should play the same role in the attraction between nuclear particles as the electromagnetic field plays in the forces between charged particles At the same time the anomalous character of the magnetic moments of neutron and proton might also be explained by the interaction with this field (Wick). The attempt was not successful, however, for two very good reasons, either of which is sufficient. First, the forces calculated are much too weak to account for the observed strength and range of nuclear forces; secondly, the theory gives attraction, in first approximation, only between neutron and proton. The forces between like particles appear in second approximation and are repulsive.

Two lines of endeavor have been pursued in trying to construct a field theory of nuclear forces. One generalized the beta interaction. This was first done by Gamow and Teller, and by Wentzel, who postulated an interaction with pure electron fields, so that neutron and proton would be able to create an electron-positron pair as well as an electron-neutrino pair. This same type of "pair emission" forces was later extended to meson pairs (Marshak) on the assumption that the meson has spin $\frac{1}{2}\hbar$. On account of the flexibility in introducing new fields these theories are able to account for the nuclear forces. There are several objectionable features of the results, however. In the electron-positron pair theory, for example, special forms of the states into which the electrons are omitted have to be assumed in order to avoid scattering of slow neutrons by the atomic electrons. In the meson-pair theory the calculated cross section for scattering of cosmic ray mesons by nuclei is higher than that observed except for very weak interaction forces (and a rather long range for the forces). There is a general objection to the meson pair theory that observations show about 30% more positive mesons at sea level than negative ones. If they were created in pairs the number might be expected to be more nearly equal. Pending further experimental evidence of the spin of the meson, however, the possibility of a meson-pair field theory cannot be ruled out.

The other class of attempts to describe nuclear forces by a field theory is typified by more direct generalizations of electromagnetic theory. The first theory of this class was presented by Yukawa, in 1935, before the meson was discovered. Yukawa found that the range and strength of nuclear forces could be understood if the nuclear particles emitted "heavy quanta" having a finite rest mass of about 200 times the electron rest mass. This quantum was conceived as having Bose-statistics, the same as a light quantum, but for simplicity the spin was assumed to be zero. Furthermore the heavy quantum could have one unit of electric charge positive or negative and Yukawa suggested that the heavy quantum could disintegrate into electron and neutrino giving beta radiation.

The discovery of the real meson, of about the same mass, by Neddermeyer and Anderson stimulated interest in Yukawa's theory. Then it was proven that charged quanta of zero spin gave repulsion between neutron and proton in the ^3S deuteron. To correct this obvious defect the spin of the quantum was assumed to be unity. The free space wave equations of the meson of spin one are, therefore, very similar to those of a light quantum, i.e. Maxwell's equations. Thus there will be the vector field quantities $\underline{E}$ and $\underline{H}$ and potentials for these fields φ and $\underline{A}$. The inclusion of a finite rest mass, however, introduces a characteristic length which we shall denote by $1/K$ where

$$K = \mu c/\hbar \qquad 1/K = 2.18 \times 10^{-13} \text{ cm. for } \mu = 177m$$

This length may be used to generalize Maxwell's equations for free space to

$$\text{div } H = 0 \qquad \text{div } E + K^2 \varphi = 0$$

$$\text{curl } E + \partial H/c\,\partial t = 0 \qquad \text{curl } H - \partial E/c\,\partial t + K^2 A = 0$$

with the usual relations

$$\underline{E} = -\text{grad } \varphi - 2\underline{A}/c\,\partial t$$

$$\underline{H} = \text{curl } \underline{A}$$

$$\text{div } \underline{A} + c\,\partial\varphi/c\,\partial t = 0$$

The condition governing the generalization is, of course, that the form of the equations remain invariant to proper Lorentz transformations and reflections and that charge be conserved. The equations for the potentials are then

$$\nabla^2\varphi - \partial^2\psi/c^2\partial t^2 - K^2\varphi = 0$$

$$\nabla^2\underline{A} - \partial^2 A/c^2\,\partial t^2 - K^2\underline{A} = 0$$

The equations may then be further modified to represent the effect of nucleons on the field. In general we may introduce a scalar quantity analogous to electric density, a vector "current density", and a vector "magnetic moment". Since the nucleons can be treated unrelativistically in good approximation we shall consider only the effective charge density, ρ, and the effective magnetic moment, $\underline{M}$. Now ρ is scalar and $\underline{M}$ is an axial vector, hence we may write

$$\nabla^2\phi - \frac{\partial^2\phi}{\partial^2 t^2} - K^2\phi = -4r\rho$$

$$\nabla^2\underline{A} - \frac{\partial^2\underline{A}}{\partial^2 t^2} - K^2\underline{A} = 4\pi/K \text{ curl } \underline{M}$$

Time independent solutions for ϕ and A in the absence of nucleons, and having spherical symmetry are proportional to $(1/r)e^{-kr}$.

Time independent solutions with ρ and $\underline{M}$ are therefore

$$\phi(\underline{r}) = \int d\tau^1 \frac{p(\underline{r}^1)}{|r - r^1|} e^{-K|r-r^1|}$$

$$\underline{A}(\underline{r}) = -(1/K) \int d\tau^1 \frac{\text{curl}\ \underline{M}(\underline{r}^1)}{|r - r^1|} e^{-K|r-r^1|}$$

For the case of point charges and moments, therefore

$$\phi(r) = (g/r)e^{-K/r} \qquad \underline{A}(r) = -(f/K)\ \text{curl}(\underline{S}(e^{-r})/r)$$

where f and g are constants representing two different "nuclear charges", one for a spin independent interaction and one for a spin dependent interaction. Let the g_1 and g_2 be the "charge" on nucleus 1 and 2 and let $(f_1/K)\underline{S}_1$ and $(f_2/K)\underline{S}_2$ be their respective moments; then if the nucleons are separated by a finite distance, r, the potential energy due to the meson field is

$$V(r) = (g_1\ g_2)/r)e^{-Kr} - \left((f_1 f_2/K^2\right)\ \underline{S}_2 \cdot \text{curl curl}\ (\underline{S}_1 \frac{e^{-Kr}}{r})$$

Differentiating out the second term we obtain

$$V(r) = \left[(g_1 g_2)/r\right] e^{-Kr} + (2/3)\ \left[(f_1 f_2)/r\right]\ e^{-Kr}(\underline{S}_1 \cdot \underline{S}_2) + \left[(f_1 f_2)/r\right] e^{-Kr}$$

$$(1/K^2 r^2) + (1/Kr) + (1/3)\ \left[\underline{S}_1 \cdot \underline{S}_2 - \frac{3\ \underline{\ }_1 \cdot \underline{r}\ \underline{S}_2 \cdot \underline{r}}{r^2}\right]$$

The first term in V(r) is the analog of electrostatic energy in the Maxwell theory.. The principal difference is the occurrence of the exponential which essentially limits the potential to a region of radius 1/K. Thus 1/K is the "range of forces" between nucleons. Similarly, the third term is the analog of magnetic dipole interact, as is readily shown by letting f and e approach zero in a constant proportion. This latter process shows that there is no analog of the second term in Maxwell's theory.

Three different assumptions about the true electric charge on the mesons are generally considered. These are (1) uncharged, (2) either positive or negative ($+e$), (3) both charged and uncharged. The theories developed on these assumptions are referred to as neutral, charged and symmetrical theories respectively. The requirements on the g's and f's for neutron and proton are somewhat different for the different theories.

It is necessary to account for equality of neutron-neutron, proton-proton, and neutron-proton forces in the singlet state. In the neutral theory, which does not distinguish in any other way between neutron and proton we have $g_1 = g_2$ and $f_1 = f_2$ for any nucleon. Consequently the potential function for a singlet: $\sigma_1 \cdot \sigma_2 = .3$, and the third term vanishes, is

$$V_e(r) = (g^2 - 2f^2)(e^{-Kr}/r)$$

The potential function for the triplet state is

$$V_1(r) = (g^2 + 2/3\, f^2)(e^{-Kr}/r) + \text{dipole-dipole term.}$$

The static, g^2, term is repulsive for both states and must be overcome by the $-2f^2$ in the singlet and by the dipole-dipole term in the triplet. In the triplet the repulsion is increased by $2/3\, f^2$ term. Bethe has shown that g may be taken to vanish and the singlet and triplet binding energies of the deuteron correctly obtained by choosing f^2 alone. The nature of the effect of the dipole-dipole term has been discussed under the theory of the deuteron. In that discussion an attractive static potential was assumed, whereas the neutral theory does not permit that assumption but rather makes up for it by the $\sigma_1 \cdot \sigma_2$ term. Bethe finds agreement with observations on the deuteron with $f = 3.24e$.

In the charged meson theory there are no neutral mesons postulated and the meson must have either $\pm e$. A neutron can emit only a negative meson and turn into a proton, and a proton can emit only a positive meson turning into a neutron. This has an important effect on the nature of the forces. The neutron and the proton are considered to be two states of the same particle. The property of being one or the other is called the isotopic spin in analogy to the ordinary spin. A nucleon may have $m_\tau = -\frac{1}{2}$ in which case it is a proton or $m_\tau = \frac{1}{2}$ for a neutron. This is directly analogous to $m_s = -\frac{1}{2}$ and $m_s = \frac{1}{2}$ for the two possible magnetic quantum numbers of the nucleon or electron. The significance of the formulation of isotopic spin is evidenced in constructing many-particle wave functions in accordance with the Pauli exclusion principle. By this principle, and the assumption that neutron and proton represent two states of the same particle, the wave function of the deuteron, for example, must be antisymmetric. Thus the ^3S state of deuteron, which is obviously symmetric in spin and coordinate space, must be antisymmetric in isotopic spin space; this means that if the two particles exchange charge, as in the emission and absorption of charged mesons, the complete wave function of the ^3S changes sign on each exchange. On the other hand the complete wave function of the ^1S will not change sign, since it is already antisymmetric due to the spin state and the isotopic spin function must be symmetric.

The nature of charge exchange is therefore manifested in the two particle potential function by giving a plus sign to states with even S + L and a minus sign to states with odd S + L. The static forces are thus attractive for some states and repulsive for others, instead of being always repulsive as in the neutral theory. Unfortunately the charged theory gives no forces between like particles, because of the impossibility of exchange in that case and cannot be seriously considered as a theory of nuclear forces. It represents the simplest case of exchange forces, however, and we shall need the description of such forces in the symmetrical theory.

The symmetrical theory postulates that both charged and uncharged mesons are represented by the field. Since the charged part contributes nothing to the forces between like particles the choice of $g_n = g_p$ and $f_n = f_p$ for the neutral mesons is ruled out for, then the forces between unlike particles and between like particles in the ^1S would be equal because of the neutral mesons alone. The additional charged field between unlike particles would then make them unequal. The potentials in the singlet neutron-neutron and neutron-proton states are

$$V_o\,(n,n) = (g_n^{\,2} - 2f_n^{\,2})(e^{-Kr}/r)$$

$$V_o(n,p) = (g_n g_p - 2f_n f_p) + 2(g_n^{\,1}\, g_p^{\,1} - 2f_n^{\,1} f_p^{\,1}) \quad (e^{-Kr}/r)$$

where $g_n^{\,1}$, $f_n^{\,1}$, etc., apply to charged mesons. The factor 2 in the primed expression takes account of the two kinds of charged mesons. The unprimed g and f refer to neutral mesons, as before. We can equate V_o (n,n), V_o (p,p) and V_o(n,p) by taking

$$g_n = -g_p \qquad g_n^{\,1}\ \ g_p^{\,1} = g_n^{\,2} = g_p^{\,2}$$

$$f_n = -f_p \qquad f_n^{\,1}\, f_p^{\,1} = f_n^{\,2} = f_p^{\,2}$$

which has the symmetrical solution: all $|g|$ are equal and all $|f|$ are equal. Only the phases of these numbers may differ.

The relative merits of the neutral and symmetrical theories have been determined by Bethe to whom this semi-classical description of nuclear fields is also due. Bethe finds that the neutral theory with $g = o$ and $f = 3.24e$ can account for the two body problems satisfactorily. There is one artifice necessary, however, and this arises because of the divergence of the dipole-dipole forces at $r = 0$. As the wave function of a particle is concentrated in a region of radius r the minimum kinetic energy is of the order $K^2/2mr^2$. The average dipole-dipole potential, on the other hand, is proportional to $1/r^3$. This shows that the smaller the radius of the wave function the lower the energy and consequently there is no finite state of lowest energy. To remedy this it is customary to assume that the $1/r^3$ law of attraction ceases at a certain "cut off" radius. In the neutral theory Bethe finds it necessary to cut off at 0.32 the range of force $(.32/\kappa)$ if it is assumed that the potential remains constant at smaller r and equal to the value at cut off. The theory then gives the correct quadrupole moment and also the correct S-states and S-scattering.

The same method applied to the symmetrical theory leads to a cut off at (1.3) $(1/\kappa)$ and to a quadrupole moment of the wrong sign. Thus the numerical results, at least for the S and D states, are highly unsatisfactory. Qualitatively the symmetrical theory is to be preferred because the charged mesons appear in cosmic rays; they might reasonably be supposed to disintegrate into electron and neutrino; and they make possible, in principle, an understanding of the anomolous magnetic moments of neutron and proton. In case the mesons are supposed to emit beta particles the Fermi constant is a product of the interaction constants pertaining to nucleon-meson interaction and meson-beta interactions. The decay period of a free meson, which has been accurately measured by Rossi, should depend only on the meson-beta interaction constant. There are, therefore, three determinations of the two constants and they are in violent disagreement. In addition to all this the calculated burst producing cross section of charged mesons in the atmosphere is much larger than the observed cross section.

From a quantitative point of view the neutral meson theory is the most promising analog of the electromagnetic theory. Perhaps part of the reason for this is that it need not explain the properties of the observed, charged mesons. There is some indication from the scattering of high energy neutrons by protons, i.e., the case in which the P-wave becomes important, that the neutral theory predicts a too large cross section. The symmetrical theory accounts for the experiments on fast neutrons satisfactorily.

In conclusion it can only be said that it is hoped that further work on scattering cross sections at higher energies, angular distributions in scattering and photoelectric processes, the nature, particularly the spin, of the mesons, etc., will guide us in formulating a field theory of nuclear forces that will explain everything.

January 13, 1944

LECTURE SERIES ON NUCLEAR PHYSICS

Fifth Series: The Statistical Theory of Nuclear Reactions

Lecturer: V. F. Weisskopf

LECTURE XXXI. Introduction

The calculation and the theoretical prediction of the properties of atomic nuclei is made difficult chiefly by two reasons: our ignorance as to the exact nature of the nuclear forces, and our inability to solve the complicated, many body problems which we face in the quantum mechanical treatment of almost any nucleus except the deuteron. Some attempts have been made, however, to obtain qualitative results from the theory in order to interpret the vast experimental material on nuclear reactions which been collected so far. In this article an account is given of the statistical methods of describing the behavior of nuclei. These methods use as few as possible actual assumptions concerning the nuclear forces or the nuclear structure. Their main assumptions may be summarized in the following points:

A) The nucleus can be considered as a "condensed phase" of the neutron-proton system in the thermodynamical sense. The neutrons and protons form a state in which they are densely packed, so that the nuclear matter has definite boundaries: the volume is proportional to the number of constituents. By assuming a spherical form, one can, therefore, introduce a nuclear radius $R = r_0 A^{1/3}$ where A is the number of constituents and $r_0 = 1.45 \times 10^{-13}$ cm. The results from this formula are in fair agreement with values from experiments in which the size of the nucleus is involved. Furthermore — and this is the main part of assumption A — the characteristic properties of nuclear matter, especially the close proximity of all constituents, are maintained even if the nucleus is excited to energies high enough so that it might emit one or more constituents. This is valid because the probability of emission of the constituents is very small, so that the nucleus is in a well defined state before having emitted the particle. It is in a state which has essentially the same properties as the lower excited states which do not emit particles.

This is in definite contrast to the situation with excited atoms. An electron escapes very rapidly if excited above ionization energy and no states of the atoms in which the electron is still close to the atom could possible be defined in this region of excitation. The nuclear conditions however, are similar to that of a liquid drop or a solid body, since the heat energy of such systems may well be much higher than the work necessary to evaporate one single molecule. The state before emission of this molecule has a life time long enough to be well defined. Emission of a constituent by means of nuclear excitation may then be divided into two steps: first the excitation to the excited state, and then the emission of the particle from the excited state. The analogous process in atoms, however, must be described in one single step, since the time interval until the electron leaves the atom is not long enough to define a regular quantum state in which the electron is still within the atom.

The assumption A is no longer fulfilled for very high excitation energies of the nucleus. The range of energy in which the nucleus can be considered as in the condensed phase varies from nucleus to nucleus, but it is safe to assume that the assumption is true for a range of excitation up to a large fraction of its total binding energy.

B) According to Bohr, a nuclear reaction can be divided into two well separated states. The first is the formation by the incident particle of a compound nucleus in a well defined state; the second is the disintegration of the compound system into the product nucleus and an emitted particle. The second stage can be treated as independent of the first stage of the process. The basis of this assumption is the classical picture of a nucleus as a system of particles with very strong interaction and short range forces. If the incident particle comes within the range of the forces, its energy is quickly shared among all constituents well before any reemission can occur. The state of the compound nucleus is now no longer dependent on the way it was formed and could have been produced by any other particle with corresponding energy incident on a suitable nucleus. Its decay into an emitted particle and a residual nucleus is thus independent of what happened before.

Bohr's assumption is certainly right for a case in which there is only one single quantum state of the compound nucleus which can be formed by the incident particle considering its energy and other circumstances. Then, evidently, the properties of this state are well defined and independent of the way it has been excited. It is very probably not a good assumption if the state of the compound nucleus which is formed by the incident particles consists of a superposition of several stationary states. The course of the process then depends strongly on the relative phases of the states, and is therefore not independent of the initial process of excitation. In the case, however, that the density of states of the compound nucleus is very high, so that their widths overlap each other strongly, a great many states can be excited simultaneously by the incident particle. This phase relation may be at random and the resulting processes, therefore, independent of the way of excitation. It is possible that Bohr's assumption regains approximate validity for this case. This is made more than probable by the classical picture of strong binding forces used above. Actually, classical considerations are justified in the region of high level density with overlapping levels where no quantum selectivity occurs.

C) The statistical method assumes the existence of average values of certain magnitudes averaged over states within a not too wide excitation energy interval. The averages are supposed to be slowly varying functions of that energy. The magnitudes concerned here are level distances, transition probabilities to certain states, etc. The validity of this assumption can only be proved by successful applications. If the energy intervals of averaging must be taken too wide in order to assure slowly varying functions for the average values, the statistical method loses most of its value.

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LECTURE SERIES ON NUCLEAR PHYSICS

Fifth Series: The Statistical Theory of Nuclear Reactions

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LECTURE XXXII. Qualitative Treatment

A. The Spectrum of a Nucleus

The general character of the energy states of a nucleus can be described in the following manner. There is, so far, no definite regularity found established in the spacing of the energy levels. The distance between the lowest levels is of the order of magnitude of 1 Mev. Sometimes they lie closer and in groups. Information about lowest levels can be obtained mainly from γ-ray spectra. Recently measurements of the energy loss of scattered particles have made it possible to determine the nuclear levels directly. It is interesting to note that there is no indication of a marked dependence of the average distance of lowest levels on the weight, except a slight tendency to smaller distances for higher weight.

It is safe to assume that the level density increases with higher excitation energy as is expected of almost any mechanical system. This increase should be the stronger the more particles the nucleus consists of. There is, however, no experimental material available for determining l vel distances above 2 or 3 Mev. Only the slow neutron capture experiments provide some very scant evidence on the level density around 8 Mev excitation as is explained below.

The states of the nucleus, with the exception of the ground state, have a finite lifetime because of the possibility of radiative transitions to lower states and of ejection of particles. (The emission of β-rays can be neglected here since its probability is extremely small.) The states thus have a finite width Δ^n which is connected with the lifetime τ_n of the nth state according to the relation $\Delta^n = \hbar/\tau_n$. If the finite lifetime is due to different emissions, the total width Δ^n is the sum of partial widths, which themselves are proportional to the specific emission probabilities. We therefore can write:

$$\Delta^n = \sum_a \Gamma_a^n + \Gamma_r^n$$

where Γ_r^n is the radiation width and Γ_a^n the partial width corresponding to the emission of a particle a, which may be a proton, a neutron, or an α-particle, etc.; $\Gamma_a^n/\hbar$ is the emission probability per second of a particle a by the state n.

The width of the low lying levels is purely a radiation width as long as the excitation energy is lower than the lowest binding energy of a particle. The binding energy B_a of a particle a is the energy necessary to remove it from the ground state and leave the residual nucleus in its ground state. We therefore get $\Gamma_a^n = 0$ if $E_n < B_a$, E_n being the excitation energy of the nth level above the ground state. The binding energy B_a of a proton or a neutron is found to be about 8 Mev for nuclei which are not too near the lower or upper end of the periodic table. The very light elements about up to oxygen do not show any regularity. From then on, 8 Mev is a good average but the value may be several Mev higher or lower in individual cases. At the heavy end of the table ($A > 200$) the binding energy is smaller and reaches an average of 6 Mev for the elements of the weight of U.

These figures are taken from the total binding energy W_0 of the nucleus; this is the energy necessary to decompose the nucleus into neutrons and protons, which is given by the mass defect. The total binding energy W_0 is roughly proportional to the number A of constituents and it is therefore justified to assume that the binding energy B_a of one particle is W_o/A, which leads to the figures mentioned above.

We know, however, from the Q-values of p-n reactions that it requires sometimes as much as 4 Mev to replace a neutron by a proton. This shows that large fluctuations in B_a from the average value must be expected.

With sufficient excitation energy E_n, a particle a can be emitted with different energies, corresponding to the different states in which the residual nucleus can be left behind. We then can subdivide Γ_a^n into different partial widths:

$$\Gamma_a^n = \sum \Gamma_{a\alpha}^n, \tag{1}$$

where $\Gamma_{a\alpha}^n$ denotes the width corresponding to an emission of a with the special condition that the residual nucleus should be left in the state α.

The widths Δ^n increase with higher excitation energy since, firstly, more particles can be emitted and secondly, more different states α of the residual nuclei are possible, so that the number of terms in the sum (1) increases, and finally, it is expected that the probability of emission of a particle increases with its velocity. At a certain excitation energy E the width Δ^n becomes larger than the level distance and the levels overlap.

The nuclear levels in the region above $E_n = B_a$ can be investigated by the bombardment of nuclei with particles a of a given energy ϵ. If the particle hits the nucleus and is absorbed, it forms a compound nucleus with an excitation energy $B_a + \epsilon$. This absorption should be appreciable only if the energy $B_a + \epsilon$ is equal to, or within the width of, an excitation level of the compound nucleus. Thus with monoenergetic particle beams, it is possible to investigate, by varying the energy, the spectrum of the compound nucleus from B_a upwards. We expect to observe "resonance" absorption when $B_a + \epsilon$ is equal to an excitation energy E_n. Resonance levels have been observed with α-particle and proton bombardments on light elements up to aluminum. Unfortunately, the average distance of levels with an excitation energy above B_a becomes very small with increasing atomic number so that the energy of the bombarding particle beam cannot be defined sharply enough to separate them in heavier nuclei. No resonance has been observed with beams of protons, deuterons, or α-particles for elements heavier than zinc $(Z = 30)$. $A \approx 66$.

Although it is even more difficult to produce monoenergetic neutron beams of arbitrary energy, there is one energy region in which neutrons can be used with great success; namely the region of very low thermal energies. Neutrons can be slowed down until they reach thermal or nearly thermal energies. Their energy is then very well defined compared to the poor energy definition of beams of fast particles. Their energy can also be measured to a high accuracy by means of absorbers which have, in that region, a known energy dependent absorption (boron). Recently methods have been developed to produce neutron beams with very sharp energy up to 5 ev.

Resonance absorption with slow neutrons can, therefore, be expected even for heavy nuclei. Although it is not difficult to produce similar low energy beams of charged particles, they are useless because they do not penetrate the Coulomb barrier of the nucleus.

The resonance capture experiments with slow neutrons give information as to the shape and width of the resonance levels of the compound nucleus and will be discussed in connection with the quantitative expressions. It is found that the main contribution to the width of these levels is the radiation width. This means that a captured slow neutron will in all probability stay within the compound nucleus, since the most probably effect is the emission of γ-quantum after which the excitation energy sinks below B_a and no particle emission is possible.

Some evidence as to the level density of the compound nucleus can be obtained from the number of nuclei which are found to show resonance capture of slow neutrons. This number indicates the probability that a level lies in a region of a few volts above B_a. One obtains by that method a level distance of roughly of the order 10 ev at the excitation energy B_a for elements whose atomic number A is about 100 or higher. It is much larger for lighter elements; in the region of $A \sim 60$ it is probably of the order of 100 ev.

We now proceed to the discussion of the cross sections of actual nuclear reactions. According to assumption B, we may write for the cross section $\sigma(a,b)$ of a nuclear reaction in which a nucleus is bombarded with a particle a and a particle b is ejected:

$$\sigma(a,b) = \sigma_a \eta_b \qquad (2)$$

Here σ_a is the cross section for the formation of the compound nucleus, and η_b is the relative probability that the particle b is emitted.

The cross section σ_a can be written in the following form:

$$\sigma_a = S_a \xi_a$$

Here S_a is the maximum value σ_a can possibly assume and ξ is the "sticking probability" of a which necessarily is smaller than unity. For neutrons whose wave length $\lambda\!\!\!^{-}$ is large compared to the nuclear radius R $S_a = \pi \lambda\!\!\!^{-2}$. For charged particles, S_a can be roughly defined as the cross section for penetration to the surface of the nucleus and is strongly dependent on the energy in the familiar way of the Gamow-penetration of Coulomb barriers. The quantitative discussion is found in Section XXXIII.

ξ_a is the probability that a formation of the compound nucleus takes place, if the particle has reached the nucleus. As a function of the energy, ξ will display the resonance properties, mentioned before. It will be small between the resonances and large when $B_a + \epsilon$ is equal or near to an excitation level of the compound nucleus. From a certain value ϵ on, the widths of the levels overlap and ξ becomes a smooth function which, however, must be smaller or equal to unity. Naturally, an average value of ξ over the resonance is needed if the incident beam is not sharp enough in energy to separate the levels.

It is reasonable to assume that ξ becomes unity if $\lambda\!\!\!^{-}_a$ is small compared to the distance between the nuclear constituents. Then, classical considerations can be applied, which make it seem very plausible that every particle which comes within the range of the nuclear forces is incorporated into a compound nucleus. This condition is fulfilled for energies above 10 Mev.

The second factor in (2), η_b is the relative probability that a particle is emitted by the compound nucleus. η_b is evidently given by

$$\eta_b = \overline{\Gamma}_b / \Sigma_c \overline{\Gamma}_c \tag{3}$$

Here $\overline{\Gamma}_b$ is the partial width of the compound state, for the emission of a particle b, averaged over the compound states which are excited by the first stage of the process. The sum in the denominator should be extended over all particles c which are ejected.

The computation of the $\overline{\Gamma}_a$ is fundamentally not different from the calculation of the σ_a. The Γ_a are the probabilities of the opposite process to the formation of a compound nucleus and there is a fundamental relation between the probability of opposite processes. This relation is worked out in detail in Section XXXIII.

A few qualitative conclusions can be drawn without using the formulas. The Γ_n for neutron emission is always much greater than the Γ_p for proton emission, or the Γ_b for any charged particle, because of the Coulomb potential barrier, except under unusual conditions that the binding energy of the proton is considerably smaller than that one of the neutron, so that the proton has much more energy at its disposal. Thus, if a reaction with neutron emission is energetically possible, this is generally more probable than all the other emissions. exceptions are found only just above the threshold of a reaction with neutron emission, when the neutron has very small energy. The Γ_p and Γ_α are strongly energy dependent and become smaller with larger Z. Nuclear reactions with p or α-emissions therefore are very rare for heavy elements.

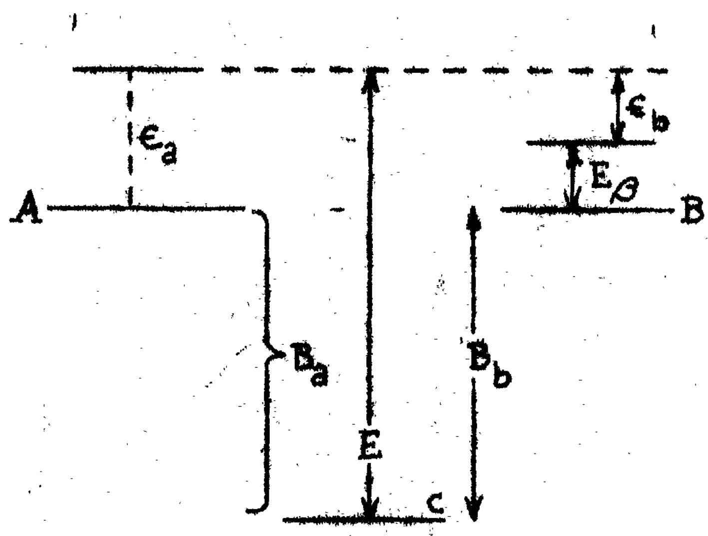
Deuteron emissions have never been observed. This is because of the high energy to remove a deuteron from the compound nucleus; it is equal to the work of removing two constituents minus the small amount of the deuteron binding energy. The emission of a neutron alone is therefore by far much more probable.

What can be predicted as to the energy distribution of the emitted particles? After emission of a particle, the residual nucleus is left in one of its excited states, say with an excitation energy E_β or in its ground state. The energy

$$\epsilon_b = E - B_b - E_\beta$$

where E is the excitation energy of the compound nucleus which is in turn determined by the energy ϵ_a of the incident particle a, producing the compound nucleus: $E = B_a + \epsilon_a$. To every level E_β of the residual nucleus corresponds and energy value ϵ_b/β. Thus the energy distribution of the particles emitted consists of a series of peaks which are a picture of the spectrum of the residual nucleus, the highest energy corresponding to the lowest level of the residual nucleus. In the special case that the incident particle is the same as the emitted one - (n,n) - (p,p) process - we thus obtain

$$\epsilon_b = \epsilon_a - E_\beta$$



A - Ground level of initial nucleus
B - Ground level of final nucleus
C - Ground level of compound nucleus

The highest energy ϵ_b corresponds to the elastic scattering. This peak will be the strongest, especially in the case of charged particles, since it contains also those particles which are scattered by the outside fields. Actually it is impossible to distinguish between the following four contributions to the elastic scattering, which give rise to scattered wave-functions which are superposed coherently*).

a) Scattering by the Coulomb field if any

b) Scattering by the shape of the nucleus in analogy to the optical diffraction

c) Penetration to the surface of the nucleus and reflection there

d) The forming of a compound nucleus with successive re-emission with equal energy

It is, however, important to distinguish d) from the other contributions in order to define the sticking probability ξ. The part $\sigma_c \lambda$ of the elastic scattering*) which corresponds to the formation of the compound nucleus can be defined by connecting it with probability $\overline{\Gamma}_{a0}$ of the inverse process of emission of the particle by the compound nucleus, with leaving the residual nucleus in its ground state "O". The latter process is completely defined and the cross section of its inverse is well defined (see Section XXXIII).

If the energy E is high enough, a great many levels E of the residual nucleus can be excited. The energy distribution of the particles emitted becomes continuous in the region where the levels of E_β are closer together than the definition of the energy of the incident beam. The shape of the distribution function $I(\epsilon_b)$ is then essentially given by the level density $\omega(E_\beta)$ of the residual nucleus, multiplied with the average emission probability $P(\epsilon_b)$ into one residual level of that energy:

$$I(\epsilon_b) = P(\epsilon_b) \cdot \omega(E_\beta), \quad E_\beta = E_r - B_b - \epsilon_b \tag{4}$$

As it is shown quantitatively in Section XXXIII, this probability $P(\epsilon_b)$ is in

*) The problem is equivalent to the discussion whether resonance fluoresence can be interpreted as pure Rayleigh scattering or whether the atom is actually lifted to the excited state and subsequently emits the scattered light. Because of the coherence of the incident and scattered light, the first interpretation is more appropriate.

general (apart from individual fluctuations) proportional to the energy ϵ_b and to the penetration probability through the potential barrier in case of charged particles. $P(\epsilon_b)$ is an increasing function, $\omega(E_\beta)$ a decreasing function of ϵ_b. We therefore obtain an energy distribution in the region in which the levels of the residual nucleus become undissolved, with a definite maximum which is in the case of uncharged particles very similar to a Maxwellian distribution. This is discussed quantitatively in Section XXXVI.

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Fifth Series: The Statistical Theory of Nuclear Reactions

Lecturer: V. F. Weisskopf

LECTURE XXXIII. Cross Sections and Emission Probabilities

The values of the cross sections of the formation of a compound nucleus by impact of a particle with a nucleus, and the probabilities of the inverse process, the emission of a particle by a nucleus, obey certain general rules, which are derived from the geometrical properties of the problem.

Let us consider the cross section $\sigma^{(c)}$ of the formation of a compound nucleus. It is appropriate to decompose any cross section σ of a nuclear process into the partial cross sections σ_l belonging to the contributions of the particles with an angular momentum $\hbar l$ in respect to the scattering center:

$$\sigma^{(c)} = \sum_l \sigma_l^{(c)} \qquad (6)$$

The angular momentum in respect to the scattering center of a classical particle moving in a straight beam is given by m v d where d is the nearest distance of a straight direction of motion to the center. Quantum theory admits only integer multiples of $\hbar$ as values for the angular momentum and it is justified to say that, in a straight beam, all particles moving along lines, whose distance d from the center lies between $l\lambda\!\!\!^{-}$ and $(l+1)\lambda\!\!\!^{-}$ ($\lambda\!\!\!^{-} = \hbar/mv$) have an angular momentum $\hbar l$. Thus all particles of angular momentum $\hbar l$ cross an area perpendicular to the beam of the size $(2l+1)\pi\lambda\!\!\!^{-2}$. A nuclear reaction which removes particles of a given angular momentum from the beam, cannot remove more than come in and thus never have a cross section σ_l larger than that area:

$$\sigma_l^{(c)} \leq (2l+1)\pi\lambda\!\!\!^{-2} \qquad (7)$$

A more exact derivation of this relation is given below.

It is to be expected that particles, whose nearest approach on a straight line, d, is larger than the range R of any forces coming from the scatterer, will not be scattered at all. Thus $\sigma_l = 0$ if d > R or

$$\sigma_l^{(c)} = 0 \qquad (l\lambda\!\!\!^{-} > R)$$

We obtain the maximum possible cross section of a system if we assume that $\sigma_l^{(c)}$ has its maximum value for $l\lambda\!\!\!^{-} < R$.

$$\sigma^{(c)} \leq \sum_0^{R/\lambda\!\!\!^{-}} (2l+1)\pi\lambda\!\!\!^{-2}$$

In case $R/\lambda\!\!\!^{-} \gg 1$ this sum gives in a good approximation

$$\sigma^{(c)} \leq \pi R^2 \qquad (R/\lambda\!\!\!^{-} \gg 1) \qquad (8)$$

a result which is not surprising since it means that the cross section of a nuclear reaction must be smaller or equal to the geometrical cross section, if the wave length is small compared to the radius.

We now proceed to the exact derivation of these results. We use the well known decomposition of a plane wave in the z-direction.

$$\psi = e^{ikz} = (2\pi/\sqrt{2kr}) \sum_{l=0}^{\infty} \sqrt{2l+1}\, i^l J_{l+\frac{1}{2}}(kr)\, Y_l^{(0)}(\vartheta) \qquad (9)$$

Here $J_{l+\frac{1}{2}}$ are Bessel functions and $Y_l^{(0)}$ are spherical harmonics normalized to unity.

$$2\pi \int (Y_l^{(0)})^2 \sin\vartheta \, d\vartheta = 1$$

In this series (9) the plane wave is decomposed into subwaves belonging to the angular momentum $\hbar l$. It is seen from the asymptotic expression at large r.

$$e^{ikz} \rightarrow \frac{\sqrt{\pi}}{kr} \sum_{\ell=0}^{\infty} \sqrt{2\ell+1}\ i^{\ell} \left[e^{-i(kr+\ell\pi/2)} - e^{+i(kr+\ell\pi/2)} \right] Y_{\ell}^{(0)} \quad (10)$$

that every subwave consists of a superposition of an incoming and an outgoing spherical wave of equal intensity (first and second term in the square bracket).

If a particle wave of this form impinges upon a nucleus, the superposition is changed in two ways: a) the intensity of the outgoing spherical wave is diminished due to absorption of the particles (the latter may or may not be reemitted); b) the phase of the remaining outgoing wave is changed. Both effects give rise to a scattered wave since the outgoing wave is no longer able to interfere with the incoming wave in the same way as in (9), the only way in which there is no radiation from the center. Thus the actual wave function ψ has not the asymptotic form (10) but

$$\psi = \frac{\sqrt{\pi}}{kr} \sum_{\ell=0}^{\infty} \sqrt{2\ell+1}\ i^{\ell} \left[e^{-i(kr+\ell\pi/2)} - \eta e^{2i\delta_\ell} e^{+i(kr+\ell\pi/2)} \right] Y_{\ell}^{(0)} \quad (10a)$$

The outgoing waves have received a factor $\eta e^{2i\delta_\ell}$ where $\eta \leq 1$. If absorption has taken place, η is smaller than unity. $2\delta_\ell$ is the phase shift of the outgoing waves.

Let us decompose (10a) into the incoming subwaves:

$$\phi_\ell^{i} = \frac{\sqrt{\pi}}{kr} \sqrt{2\ell+1}\ i^{\ell}\, e^{-i(kr+\ell\pi/2)}\, Y_\ell^{(0)} \quad (11)$$

and the outgoing ones

$$\phi_\ell^{o} = \frac{\sqrt{\pi}}{kr} \sqrt{2\ell+1}\ i^{\ell}\, \eta\, e^{2i\delta}\, e^{i(kr+\ell\pi/2)}\, Y_\ell^{(0)} \quad (11a)$$

The mount absorbed per second, for a single ℓ, is given by the difference of the currents of $\phi_\ell^{(i)}$ and $\phi_\ell^{(o)}$ through a large sphere of radius r:

$$r^2 \int \left[v|\phi^{(i)}|^2 - v|\phi^{(o)}|^2 \right] \sin\vartheta\, d\vartheta = (1-\eta^2)\, v(2\ell+1)\left(\pi/k^2\right)$$

Since there are $v = \hbar k/m$ incident particles per cm^2 and second in the original plane wave e^{ikz}, the cross section for the absorption of particles of angular momentum ℓ, is

$$\sigma_\ell^{(c)} = (1-\eta_\ell^2)\pi \lambda\!\!\!^{-2} (2\ell+1) \quad (12)$$

This is in agreement with the statement contained in (7).

The difference between (10a) and (10) is the wave which one has to add to a plane wave (e^{ikz}) to get the actual wave ψ; it is therefore the scattered wave:

$$\psi - e^{ikz} = \frac{\sqrt{\pi}}{kr} \sum_{l=0}^{\infty} \sqrt{2l+1}\; i^{l}(1-\eta e^{2i\delta})\, e^{i(kr+l\pi/2)}\, Y_l^{(0)} \qquad (13)$$

The cross section $\sigma_l^{(el)}$ for elastic scattering, for a single l, can be obtained by calculating the current of the lth subwave of $\psi - e^{ikz}$ through a large sphere and dividing it by the current of the incident wave. This gives

$$\sigma_l^{(el)} = \left|(1-\eta e^{2i\delta})\right|^2 \pi \lambda\!\!\!^-{}^2 (2l+1)$$

This can also be written in the form

$$\sigma_l^{(el)} = \left[(1-\eta)^2 + 4\eta \sin^2 \vartheta\right](2l+1)\pi \lambda\!\!\!^-{}^2 \qquad (14)$$

It is seen from (12) and (14) that certain relations hold between the capture cross section $\sigma_l^{(c)}$ and the elastic one, $\sigma_l^{(el)}$. The upper limit of the capture cross section is given by (7). The elastic cross section, however, has a higher upper limit, which is reached if $\eta = 1$ and $\vartheta_l = \pm \pi/2$:

$$\sigma_l^{(el)} \leq (2l+1)\, 4\pi \lambda\!\!\!^-{}^2$$

This maximum can only be reached if $\eta = 1$ or if $\sigma_l^{(c)} = 0$

This can also be understood in the following way: the maximum effect of elastic scattering is obtained by a change of phase of π of the second term in the square bracket of (10) (the outgoing subwave) without reducing its intensity. This is identical with adding to the plane wave just twice the outgoing subwave. Thus the cross section corresponding to this, is 2^2 times larger than the one resulting from absorbing the outgoing subwave. Hence we get a maximum cross section for scattering four times larger than for absorption.

The expressions (12) and (14) limit the possible values of $\sigma_l^{(el)}$ for a given value of $\sigma_l^{(c)}$. This is indicated in Fig. 1, where the upper and lower limits of $\sigma_l^{(el)}$ is plotted against $\sigma_l^{(c)}$. It is seen that it is necessarily $\sigma_l^{(el)} = \sigma_l^{(c)}$ if $\sigma_l^{(c)}$ assumes its maximum value $(2l+1)\pi \lambda\!\!\!^-{}^2$.

The following theorem is of interest, since it connects the total cross section with the amplitude of the elastic scattering: the total cross section is given by

$$\sigma_\ell^{(tot)} = \sigma_\ell^{(c)} + \sigma_\ell^{(el)} = (2 - 2\eta\cos 2\delta_\ell)\pi \lambda\!\!\!^{-2}(2\ell+1)$$

which also can be written in the form:

$$\sigma_\ell^{(tot)} = (A_\ell + A_\ell^*)\pi \lambda\!\!\!^{-2}(2\ell+1)$$

where $A = (1 - \eta e^{2i\delta_\ell})$ is the value by which one has to multiply the outgoing subwave of the undisturbed plane wave in order to get the scattered subwave(13).

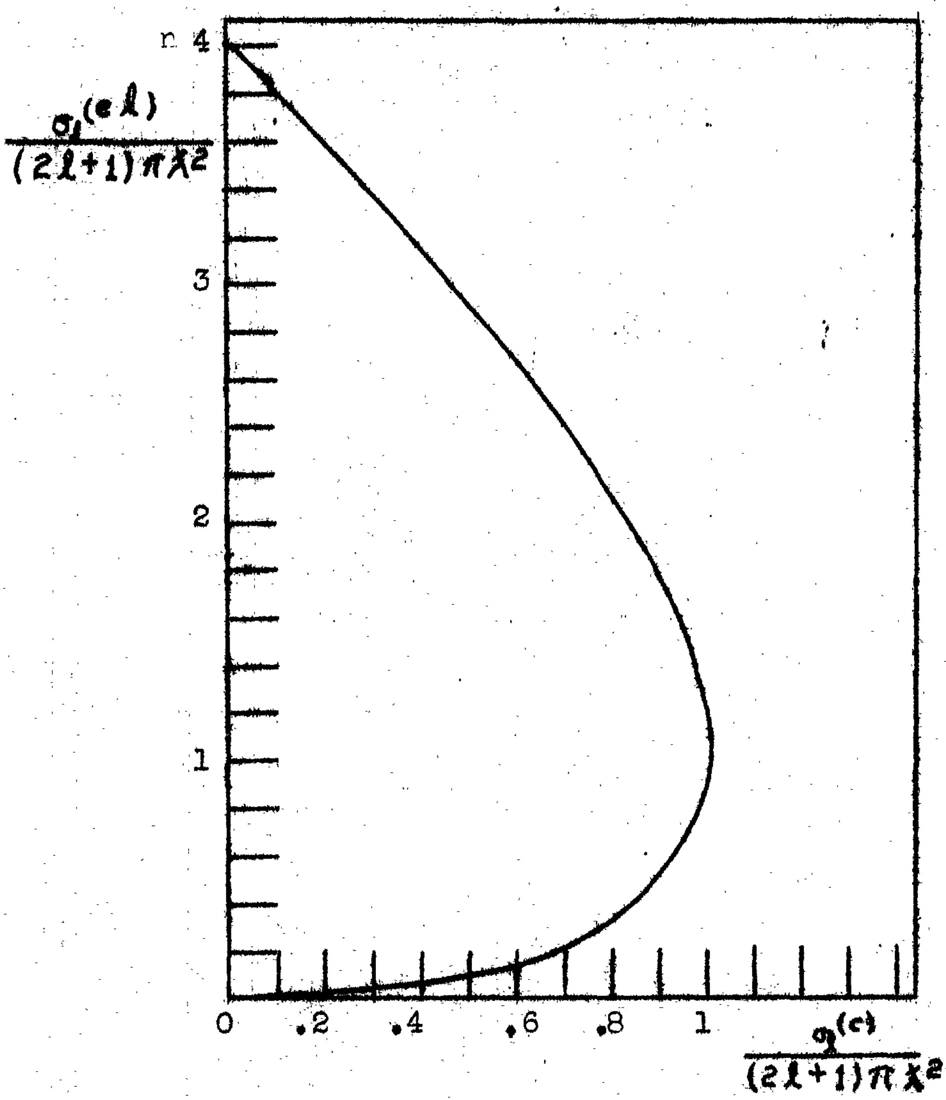

Fig. 1: Upper and lower limit of the elastic cross section for given capture cross section.

If the scattering object (actually the range of the scattering forces) is large compared $\lambda\!\!\!^{-}$, all waves for which $\ell\lambda\!\!\!^{-} < R$ can be absorbed, but the waves for which $\ell - R/\lambda\!\!\!^{-} \gg 1$ vanish for $r \gtrsim R$ and thus are not influenced and should have $\sigma_\ell^{(c)} = 0$. If $R/\lambda\!\!\!^{-}$ is a large number, it is a good approximation to assume $\sigma_l = 0$ if $l\bar{\lambda} > R$, and we obtain the expression (6) for $\sigma^{(c)}$.

It is interesting to note, that the total cross section $\sigma_{total} = (\sigma^{(el)} + \sigma^{(c)})$ of a spherical object, large compared to the wave length of the incident beam, is <u>twice</u> the geometrical cross section πR^2 if it totally absorbs all particles that hit it. This is seen in the following way: our assumption of total absorption is equivalent with the statement that $\sigma^{(c)}$ reaches its maximum value πR^2. According to Fig. 1 the elastic cross section is then equal to $\sigma^{(c)}$, so that $\sigma^{(tot)} = 2\pi R^2$. This result, paradoxical at first, is explained by the fact that a sphere which absorbs all particles falling upon it, casts a shadow in the beam. A shadow is, in terms of the wave picture, a peculiar interference effect of a scattered wave with the incident wave. Evidently the scattered wave producing this effect, must have an intensity equal to the amount of beam taken out by the shadow. Thus we get also an elastic cross section of πR^2. The shadow becomes diffuse in a distance $\sim(R^2/\bar{\lambda}$ from the object, since the waves passing by the object are diffracted and change their direction by an angle $\sim(\bar{\lambda}/R)$. In distances large compared to $R^2/\bar{\lambda}$ the shadow is dissolved and an elastic scattering can be observed within a scattering angle of $\bar{\lambda}/R$.

These considerations are valid only if $\bar{\lambda} \ll R$. If $\bar{\lambda}$ becomes comparable to R, the elastic scattering is essentially unpredictable, since then assumes its maximum value only for very few values of l and it is mostly lower than its maximum.

The geometrical limitations of cross sections are of special importance in connection with the relation between inverse processes. Any limitation on a cross section of formation of a compound nucleus, also implies a limitation of the probability of emission of a particle by the compound nucleus. This connection is based upon the principle of detailed balance which connects processes which are inverse to one another.

We compare the following two processes: a) the formation of a compound nucleus by a particle a, with the angular momentum $\hbar l$, incident upon an initial nucleus in the state α; b) the emission by the compound nucleus, of a particle a with an angular momentum $\hbar \ell$, and leaving the residual nucleus in the state α. The two processes a) and b) are inverse. The first process is measured by a cross section $\sigma_{a\alpha\ell}$ and the second by an emission probability $\Gamma_{a\alpha\ell}$ We should also have specified the direction of the spin of the emitted particle in respect to some arbitrary axis. We omit this and assume in the following that the spin is given together with the angular momentum ℓ. In the definition of $\sigma_{a\alpha\ell}$ it is assumed that the incoming beam is spread over an energy interval large enough to excite many component levels. $\Gamma_{a\alpha\ell}$ is the average value of the emission probability, averaged over these levels which can be excited by the first process within a small energy region.

In order to save indices, we replace the triple index $(a\alpha\ell)$ by α and write σ_α and Γ_α , wherever it is practicable. The result of the following considerations will be the relation

$$\Gamma_\alpha = \frac{\sigma_\alpha D}{2\pi^2 \lambda^2}$$

where D is the average distance between the levels, which can be excited. Before going into the detailed derivation, it is useful to understand, why a relation of this character is expected to exist.

Let us first assume that no other process but the one characterized by Γ_α can follow the absorption given by σ_α. In this case, the level width is equal to Γ_α. It is to be expected that σ_α is proportional to the ratio Γ_α / D of the width to the distance of levels, because this magnitude measures the relative portions of the spectrum in which absorption is possible. It is plausible that the maximum absorption is reached, if Γ_α / D is of the order of unity. The maximum value of σ_α is $\pi \lambda^2 (2\ell+1)$ and we may expect a relation of the type

$$\frac{\sigma_\alpha}{\pi \lambda^2 (2\ell+1)} = \text{const.} \frac{\Gamma_\alpha}{D}$$

where the constant is of the order unity. The detailed calculation shows that it is $2\pi/(2\ell+1)$. This consideration also holds true if the compound nucleus created by the absorption of the particle a can decay in several ways. This effect would broaden the lines but would not change the absorption average over the lines.

Let us now compare the probabilities of the two inverse processes. The probabilities should be equal if they ended in states of equal weight. Since this is not the case, they should be equal after dividing them by the statistical weight of the end states.

The probability W_1 per second of the creation of the compound nucleus by the particle a if it is enclosed in a big sphere of radius R is given by

$$W_1 = \frac{\sigma_a}{(2\ell+1)\pi \lambda_a^2} \frac{v}{2R} \tag{15}$$

where v is the velocity of the particle and λ its wave length. This can be understood in the following way: if σ_α assumes its maximum value $(2\ell+1)\pi\lambda_a^2$, the particle with an angular momentum $\hbar\ell$ is sure to be captured on its way to the center of the big sphere. The average probability per unit time is then v/2R since it may be found anywhere along a diameter. (Its angular momentum $\ell\hbar$ being fixed, it moves on lines which have a given distance $\ell\lambda$ from the center. The length of these lines is nearly 2R, since R is assumed very large: $R \gg \ell\lambda$) In case σ_α is smaller than its maximum value this probability is reduced in the corresponding ratio and (15) results.

We now compare the statistical weights of the end states of the two inverse processes. If the energy of the incident particles lies between ϵ and $\epsilon + d\epsilon$ the statistical weight of the end state is ω_c(E) dE, where ω_c(E) is the level density of the compound nucleus at the excitation energy E which will be excited. (Only these levels are counted, which can be excited owing to angular selection rules.)

The probability of the inverse process is $\Gamma_\alpha/\hbar$ and the statistical weight of its end state is the statistical weight of a free particle with the energy between ϵ and $\epsilon + d\epsilon$ and the angular momentum $\hbar\ell$ in a sphere of a radius R, namely $\left[(2\ell+1)/\pi\hbar v\right] R\, d\epsilon$

We thus get

$$\frac{\Gamma_a/\hbar v}{(2\ell+1)R} = \frac{W_1}{\omega_c(E)}$$

and from (15)
$$\Gamma_\alpha = \frac{\sigma_\alpha}{2\pi^2 \lambda_a^2} \frac{1}{\omega_c(E)} \tag{16}$$

By expressing cross section σ_α by a dimensionless magnitude $Q_\alpha = \sigma_\alpha / (2l+1)\pi\lambda_a^2$ which never can be larger than unity, and by introducing the average level distance $D = 1/\omega_c(E)$ of the compound nucleus, we obtain
$$\frac{\Gamma_\alpha}{D} = Q_\alpha \frac{(2l+1)}{2\pi} \tag{16a}$$

The partial width of the emission of a particle with a given angular momentum and with the residual nucleus left in a definite state α, is thus bound by an upper limit ($Q \leq 1$):
$$\Gamma_\alpha / D \leq (2l+1)/2\pi \tag{16b}$$

A few remarks are necessary concerning the significance of the level distance D. D is defined as the distance between the levels of the compound nucleus, from which the particle can be emitted. It is the distance between the levels which can be excited by the particle a with the angular momentum $\hbar l$ (and given spin), hitting a nucleus in the state α. On the other hand, Γ_α is the partial width for the emission of a, with the same properties averaged over the levels described above (whose average distance is D) within a certain small energy region. Expression (16a) is not bound to this definition of D. For example, D could be defined as the average distance between <u>all</u> levels of the component nucleus in this energy region; if Γ_α is then understood as the average over all these levels, (for most of which $\Gamma_\alpha = 0$), the relation (16a) is then again valid. In the following we will, however understand by D and Γ_α the average distance and width of the levels which can emit the particle a with the properties specified above.

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LECTURE SERIES ON NUCLEAR PHYSICS

Fifth Series: The Statistical Theory of Nuclear Reactions

Lecturer: V. F. Weisskopf

LECTURE XXXIV. The Excited Nucleus

Let us consider a nucleus in an excited state n in which it is able to emit particles. The state n has a finite lifetime and the probability of finding the nucleus in this state decays according to the law $e^{-\Gamma^n t}$. Γ^n is the width of the state. In general the state n can emit different particles a and these particles may be emitted with different angular momenta $l\hbar$ and the residual nucleus may be left behind in different states α. Therefore the width Γ^n can be subdivided in the following way:

$$\Gamma^n = \sum_{a\alpha l} \Gamma^n_{a\alpha l} \tag{18}$$

where $\Gamma^n_{a\alpha l}$ is the partial width for the emission of a with an angular momentum $\hbar l$, leaving the residual nucleus in the state α.

The value of $\Gamma^n_{a\alpha l}$ is determined by intranuclear and extranuclear factors, and it is useful to separate them into the following ones: the probability per second, $\Gamma^n_{a\alpha l}$ of emission of a particle a is proportional to the following magnitudes:

1) The final velocity v^n_α which it acquires after having left the nucleus and after having penetrated the potential barrier if there is any;

2) The penetrability $T_{al}(\epsilon)$ of the barrier which is defined as the relative decrease of intensity of an outgoing spherical wave, corresponding to the particle a, from the nuclear surface through the potential barrier to the outside. It is useful to consider as potential barrier, not only the Coulombfield in the case of charged particles, but also the potential of the centrifugal force

$\hbar^2 l(l+1)/2mr^2$ in the case of an emission of a particle with an angular momentum l. This force is described by a repulsive potential, inversely proportional to the square of the distance, and acts on a particle in the same way as the Coulomb repulsion. $T_{al}(\epsilon)$ is a function of the charge and of the angular momentum of a and of the energy ϵ, which it eventually acquires.

3) $\Gamma^n_{a\alpha l}$ is furthermore depending on the internal nuclear conditions which govern the separation of a from the nucleus and which are lumped together in a factor $G^n_{a\alpha l}$ Thus we can write:

$$\Gamma^n_{a\alpha l} = k^n_\alpha G^n_{a\alpha l} T_{al}(\epsilon^n_\alpha) \qquad (19)$$

where k^n_α is the wave number corresponding to the energy of the outgoing particle after penetration of the potential barrier, and is proportional to the velocity v^n_α.

A more quantitative derivation of (19) can be given as follows:*)

Let us consider a nucleus with A constituents in an excited state n, in which it is able to emit particles. The quantum mechanical description of a decaying state is somewhat different from the familiar form of an eigenfunction of a genuinely stationary state. We represent the mechanical system of the nucleus by a Schroedinger equation:

$$H\Psi_n = W_n \Psi_n \qquad (20)$$

where H is the Hamilton operator of the nucleus. The functions Ψ_n are the wave functions and depend on the coordinates $r_1 \ldots r_A$ of the A constituents.

W_n is the corresponding eigenvalue. The values W_n form a spectrum of discrete values from the ground state W_0 up to a certain limit $W_0 + \bar{E}$ where $\bar{E}$ is the minimum excitation energy at which a particle might escape from the nucleus. $\bar{E}$ is equal to the binding energy of the loosest bound particle. Above $\bar{E}$ the spectrum is strictly continuous since the total system, described by (20) necessarily includes all free states of the particle removed from the nucleus. Among these there are also states into which the particle could never have come after

*) The reader may continue on page 232 if he is not interested in the more quantitative derivation.

being emitted by the original nucleus as, for example, a state in which the particle has a very high angular momentum in respect to the residual nucleus. In order to describe the excited states of the nucleus above $\bar{E}$ which are defined by our assumption A, it is necessary to introduce a device to exclude the solutions of (20) which correspond to a residual nucleus plus a freely moving independent particle which is not created by an emission of the compound nucleus. This can be done by means of boundary conditions for Ψ_n, which should express the fact that, in case the state n includes free particles outside the nucleus, they must be represented by outgoing spherical waves. With these conditions the equation (20) no longer has a continuous spectrum above $\bar{E}$. It then has discrete eigenvalues which belong to quantum states of finite lifetime, because of their ability to emit particles.

Let us introduce a radius R, the nuclear radius, which is the shortest distance from the nuclear center at which practically no nuclear force acts upon a nuclear particle. (The particle might, however, at that distance be under influence of, say, the Coulomb force.) The eigenfunctions Ψ_n have the following form for $r_a > R$:

$$\Psi_n = \sum_{\alpha} \chi_{\alpha} \psi^n_{a\alpha}(\underline{r}_a)$$

Here χ_{α} are the eigenfunctions of the states α of the residual nucleus which is left behind, if the particle a is removed from the nucleus; $\psi^n_{a\alpha}$ is the wave function of the particle a outside the nucleus. $\psi^n_{a\alpha}(\underline{r}_a)$ is a solution of the following one body equation of the particle a in the space outside of the nucleus:

$$\Delta \psi^n_{a\alpha} + \left[(k^n_{a\alpha}) - (2m/\hbar^2) V_a(r_a) \right] \psi^n_{a\alpha} = 0$$

Here V_a is the potential outside of the nucleus ($V_a = zZe^2/r_a$, where z is the charge of the particle and Z is the nuclear charge); $k^n_{a\alpha}$ is the wave number at infinite distance, when $V_a \rightarrow 0$. It is

$$k^n_{a\alpha} = \left[(2m/\hbar^2)(W_n - W_{\alpha}) \right]^{\frac{1}{2}} \tag{21}$$

where W_{α} is the energy of the residual state α. We can write

$$\psi_{a\alpha}^{n} = (1/r_a) \sum_{lm} \Phi_{a\alpha l}^{n}(r_a)\, Y_l^{(m)}(\vartheta_a \varphi_a) \qquad (22)$$

Here $Y_l^{(m)}$ is the normalized spherical harmonics and $\Phi_{a\alpha l}^{n}$ depends on r_a only. The sum over lm contains only these values which are in agreement with the conservation of the total angular momentum. The vector sum of the angular moments (plus spins) of the outgoing particle a and of the residual nucleus α must be equal to the angular momentum of the decaying state n. If the total energy W_n is not high enough to allow the particle a to escape with the residual nucleus in the state α, the relation $W_n < W_\alpha$ is valid and $k_{a\alpha}^{n}$ will be imaginary according to (21). $\Phi_{a\alpha l}^{n}$ will be an exponentially decreasing function of r_a.

In order to reduce the number of indices, we write all three indices $a\alpha l$ into one, and write only one greek index α instead. We now want to express that only those states ψ_n should be considered, which correspond to decaying nuclear states. This can be done by postulating that the functions ψ_α^n should be outgoing waves only. The radial dependent part Φ_α^n of (22) should have the asymptotic form:

$$\Phi_\alpha^n = S_\alpha^n e^{ik_\alpha^n r_a} \quad \text{for} \quad r_a > r_0 \qquad (23)$$

where r_0 is a large distance at which the kinetic energy of all particles emitted is much larger than the potential energy $V_a(r_a)$ and the centrifugal force $l(l+1)\hbar^2/2mr_a^2$. The same can be expressed by boundary conditions:

$$(\partial/\partial r_a - ik_\alpha^n)\,\Phi_\alpha^n = 0 \quad \text{for} \quad r_a = r_0 \qquad (24)$$

(In (23) and (24) $k_{a\alpha}^n$ is given by (21) and that sign of the square root must be taken for which the real part of $k_{a\alpha}^n$ is positive.)

These conditions are, of course, compatible with the wave equation (20). The equation (20) with the (complex) boundary conditions (23) or (24) has solutions for which W_n is necessarily complex:

$$W_n = E_n - i\Gamma^n/2 \qquad (25)$$

A complex eigenvalue means that the state decays exponentially according to $e^{-\Gamma^n t}$, so that Γ^n is the reciprocal half life or the width of the state n.

(Actually k^n also is complex because of the fact that (21) contains the complex W_n. It expresses the fact that the outgoing wave (23) changes in intensity with r_a. It increases due to the fact that the nuclear state decays exponentially and that the parts farther away are emitted in an earlier stage. This change of intensity along the outgoing wave, however, is only a very small effect as long as the width Γ^n of the level is small compared to the energy of the emitted particle. The ratio between real and imaginary parts of $k^n_{a\alpha}$ is equal to the distance travelled by the emitted particle within the lifetime of the level n, measured in wavelengths $\lambda = (k_\alpha{}^n)^{-1}$. This ratio will be very large in all cases of interest and we neglect from now on the imaginary part of k_α^n completely except, of course, in the case $W_n < B_a$ (excitation energy less than boundary energy) in which $k_{a\alpha}^n$ is purely imaginary and $\Gamma^n = 0$.)

We now express the level width Γ^n in terms of the ϕ_α^n. The value of Γ^n is the reciprocal life time of the level n and therefore equal to the number of particles emitted by the nucleus per second. We first subdivide Γ^n in the following way:

$$\Gamma^n = \sum_{a,\alpha,\ell} \Gamma^n_{a\alpha\ell} \qquad (26)$$

where $\Gamma^n_{a\alpha\ell}$ is the partial width for the emission of a with an angular momentum ℓ, leaving the nucleus in the state α. We surround the nucleus by a sphere with the radius r_o and determine the number of particles a with the above properties crossing that sphere per second, which in turn is equal to $\Gamma^n_{a\alpha\ell}$. Thus we obtain (the index $a\alpha\ell$ is replaced by α)

$$\Gamma_\alpha^n = (4\pi\hbar^2/m_a)\, k_\alpha^n \left| \phi_\alpha^n(r_o) \right|^2 \qquad (27)$$

under the condition: $\oint |\psi_n|^2 d\tau = 1$, where $\oint d\tau$ integration over all variables in the limits $|r_a| < r_o$.*) If ψ_n is a slowly decaying state, so that the particle density between the boundary R of the nucleus and r_o is very small compared to the density inside of the nucleus, we can also normalize Ψ_n by the integral

$$\int_{|r_a| \leq R} |\Psi_n|^2 d\tau = 1$$

In the expression (27) the value of ϕ_α^n is still dependent on the field outside of the nucleus. Since the latter is well known it is useful to show its influence

*) $(\hbar/m_a)k_\alpha^n$ is the velocity of the particle at r_o $|\phi_\alpha^n(r_o)|^2$ is, in the normalization used here, r_o^2 times the probability of finding a particle a in one volume unit at r_o.

explicitly in the expression for the width. We therefore express $\left|\Phi_{\alpha}^{n}(r_0)\right|^2$ by the value of the same function at the nuclear radius R. We introduce the magnitude $T_{\alpha\ell}(\epsilon)$ which is the ratio by which the "(intensity x r^2)" diminishes in an outgoing wave of a particle a with an energy ϵ and an angular momentum $\ell\hbar$, if it penetrates the field around the nucleus from R to r_0. r_0 had been chosen outside of all fields of force so that the "(intensity x r^2)" is constant outside r_0. $T_{a\ell}(\epsilon)$ can be called the transmission coefficient of the potential barrier. It can be calculated from the wave equation of a particle with an energy ϵ and an angular momentum $\ell\hbar$ in the field outside of the nucleus, whose solution we will call $F^{\epsilon}_{a\ell}(r)$ for this purpose:

$$\left[\frac{d^2}{dr^2} - \frac{2m}{\hbar^2}(\epsilon - V_a) - \frac{\ell(\ell+1)}{r^2}\right] F^{\epsilon}_{a\ell}(r) = 0 \qquad (27a)$$

If we choose the particular solution which corresponds to an outgoing wave and has the asymptotic form

$$F \longrightarrow e^{+\left(\sqrt{2m\epsilon}/\hbar\right) r} \qquad (27b)$$

we get

$$T_{a\ell}(\epsilon) = \frac{1}{\left|F^{\epsilon}_{a\ell}(R)\right|^2} \qquad (27c)$$

T depends on the nature of the particle a (charged or not charged), its angular momentum $\hbar\ell$ and its energy $\epsilon = W_n - W_\alpha$ which it gets if it is emitted by the state n with a residual nucleus left in α.

The transmission coefficient $T_{a\ell}(\epsilon)$ for neutrons can be calculated in the following way: the functions $F^{(\epsilon)}_{a\ell}$ are given by the Bessel functions of half integer order and it is easily seen that the boundary conditions (27b) are fulfilled if

$$F_{a\ell}(r) = \sqrt{\pi y/2}\; H^{(1)}_{\ell+\frac{1}{2}}(y) \qquad y = k_a r$$

where $H^{(1)}_{\ell+\frac{1}{2}}(y)$ is the well known Hanckel function of $(\ell+\frac{1}{2})^{th}$ order. One gets:

$$\begin{aligned} T_{a\ell} &= 1 && \text{for } \ell = 0 \\ &= \frac{x^2}{1+x^2} && \text{for } \ell = 1 \\ &= \frac{x^4}{9+3x^2+x^4} && \text{for } \ell = 2 \end{aligned} \qquad (27d)$$

$$= \frac{x^6}{225+45x^2+6x^4+x^6} \quad \text{for } \ell = 3$$

$$\frac{x^{2\ell}}{(2\ell-1)^2(2\ell-3)^2} \qquad \text{for } x \ll 1$$

with $x = k_a R$. The value of $T_{a\ell}(\epsilon)$ for charged particles is difficult to compute exactly, because of the very slow convergence of the expressions for the radial eigenfunction of the Coulomb field in the region considered here. It is evident that it decreases strongly with rising nuclear charge Z and with falling energy ϵ. An approximate expression, calculated by using the W.K.B. method, is given by:

$$T_{a\ell} = \left(\frac{B_\ell - \epsilon}{\epsilon}\right)^{\frac{1}{2}} e^{-2C_\ell} \tag{28}$$

$$B_\ell = \frac{zZe^2}{R} + \frac{\hbar^2}{2mR^2}\,\ell(\ell+1)$$

$$C_\ell = \sqrt{2m}/\hbar \int_R^{r_2} (V-\epsilon)^{\frac{1}{2}}\,dr$$

$$V(r) = \frac{zZe^2}{r} + \frac{\ell(\ell+1)\hbar^2}{2mr^2}$$

Here ze is the charge of the particle a, B_ℓ is the energy barrier height for a particle with an angular momentum $\hbar\ell$. In the expression of C_ℓ, r_2 is the radius within which the potential energy V is higher than the energy ϵ of the particle. V includes the centrifugal term. The values of C_ℓ are given in Bethe's article*). These expressions do not hold if ϵ is near B_ℓ and the approximation is good only for higher nuclear charges $Z > 20$. We get finally an expression for the partial width

$$\Gamma^n_{a\alpha\ell} = \frac{4\pi\hbar^2}{m_a} k^n_{a\alpha} \left| \Phi^n_{a\alpha\ell}(R) \right|^2 T_{a\ell}(\epsilon_\alpha^{\,n}) \tag{29}$$

It is identical with expression (19), if we put

$$G^n_{a\alpha\ell} = \frac{4\pi\hbar^2}{m_a} \left| \Phi^n_{a\alpha\ell}(R) \right|^2$$

*) H. A. Bethe, Rev. Mod. Phys. 9, 163 (1937).

The factor G depends on the value of the eigenfunction Φ at the boundary of the nucleus and thus represents the influence at the intranuclear structure.

Expression (29) separates the effects of the forces outside of the nucleus by means of the factor T. The magnitude

$$k_{a\alpha}^{n} \frac{4\pi\hbar^2}{m_a} \left| \Phi_{a\alpha l}^{n}(R) \right|^2$$

could therefore be called partial width without outside potential barrier. This separation of the barrier effect and the nuclear effect is by no means complete. It is to be expected that the value of $\Phi_{a\alpha l}^{n}(R)$ also depends on the form of the wave outside of Rk since the inside and the outside part of the function must join smoothly.

Under certain conditions, however, there is good reason to expect that the values $\Phi_{\alpha}^{n}(R)$ do not depend on the behavior outside of the nucleus. Because of the strong forces inside of the nucleus, the average momentum p_i of a particle inside of the nucleus is very high and corresponds to an energy of 20 to 30 Mev. The average momentum is an indication in what space interval the wave function changes its value considerably. We define an average wave length $\lambda_i = \hbar/p_i$ and we can say that the wave function inside the nucleus changes its value by an amount of the order of its average size, if the coordinate of one particle is changed by λ_i. If the momentum p_o outside of the nuclear surface is much smaller than p_i:

$$p_i \gg p_o \tag{31}$$

the wave function inside the nucleus must be joined at the nuclear surface to a wave function, whose value changes very slowly compared to conditions inside. It is then a good approximation in the determination of the wave function inside to assume that

$$(\partial \Phi_{a\alpha l}^{n}/\partial r)_{r=R} = 0 \tag{32}$$

for all particles. The values for $\Phi_{a\alpha l}^{n}(R)$ one gets with this boundary condition are not very different from the true one. Since this condition does not contain any reference to the properties outside, the general behavior of $\left|\Phi_{\alpha}^{n}(R)\right|^2$ is to be expected the same for charged and uncharged particles and also the same for different l's.

Particularly, the average values $\overline{\left|\Phi_{\alpha}(R)\right|^2}$ over neighboring levels should be practically independent of the energy ϵ_a of the outgoing particle. It will depend only on the properties of the excited nucleus, whose excitation energy is $B_a + \epsilon_a$, an energy which is much larger than in the region of validity of (31) and does not change appreciably in that region. The average value $\overline{\Gamma}_{a\alpha l}$ of the partial width over neighboring levels, is then given by

$$\overline{\Gamma}_{a\alpha l} = C(E)\sqrt{\epsilon_a}\, T_{al}(\epsilon), \qquad C(E) = \left(\hbar/\sqrt{2m}\right)\overline{\left|\Phi_{a\alpha l}\right|^2}$$

where $C(E)$ is a function of the total excitation energy $E = B_l + \epsilon_a$ only. It does not vary strongly in an energy interval which is small compared to B_a. Furthermore, it is approximately equal for the different particles which can be emitted by the compound nucleus, and for the different values of angular momentum of these particles.

The usefulness of this approximation can be illustrated in a few examples: the average partial width for the emission of a neutron with $l = 0$ is given by:

$$\overline{\Gamma} = C(E)\sqrt{\epsilon_a} \tag{33}$$

which is essentially proportional to the square root of the outgoing neutron energy in the region in which (31) is valid.

The average cross section over neighboring levels for the capture of s-neutrons can be obtained from (16):

$$\sigma_c = \frac{2\pi^2 \lambda^2}{D} C(E)\sqrt{\epsilon_a} \sim \frac{C(E)}{v_a} \tag{34}$$

This relation shows the well known $1/v$ law for the capture cross section of slow s-neutrons ($l=0$).

The partial width for the emission of neutrons with $l \neq 0$ is given by

$$\overline{\Gamma} = C(E)\sqrt{\epsilon_a}\, T_{al}(E) = \frac{C\sqrt{\epsilon_a}\,(kR)^{2l}}{[(2l-1)^2(2l-3)^2]} \text{ for } kR \ll 1$$

It is thus proportional to $\epsilon^{\ell+\frac{1}{2}}$ for small energies. The emission probabilities of charged particles have the transmission coefficient $T_{a\ell}(\sigma)$ as the domineering factor.

The factor C appearing in the neutron widths cannot be determined but an upper limit can be set: we know from (17) that $\overline{\Gamma}/D \leq 1/2\pi$ for neutrons (ℓ=0). Since (33) should be valid up to about $\epsilon \sim 1$ Mev we obtain

$$C\sqrt{\epsilon} \leq D/2\pi \text{ for } \epsilon \sim 1 \text{ Mev}$$

The experimentally determined values (see Section 5), indicate that C is not much smaller than $D/2\pi$, if the energy units are choosen in Mev's.

So far the radiation of the nuclear states has been neglected. It introduces another additional width Γ_r^n, which we call the radiation width. The total width Δ^n of the level is then given by

$$\Delta^n = \Gamma^n + \Gamma_r^n \tag{35}$$

The radiation width consists of a sum of partial widths $\Gamma_{r\rho}^n$ corresponding to transitions to special levels ρ below n:

$$\Gamma_r^n = \sum_\rho \Gamma_{r\rho}^n \tag{36}$$

$U_{r\rho}^n$ is the matrix element of the optical transition from n to ρ.

The spectrum of the nucleus can be divided a) into a "stable" region from 0 to $\bar{E}$ where the levels emit light quanta only and the width is merely due to radiation; b) in a "resonance" region above $\bar{E}$ where the states are able to emit particles, but still are apart from each other by more than their width so that they form a discrete spectrum. The widths Γ^n increase with higher excitation energy. There are more partial widths contributing to the sum in (18) and every single partial width increases with higher energy of the particle emitted. We so obtain a third region, the "continuous" region, in which the level width is equal to or larger than the level spacing.

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LECTURE SERIES ON NUCLEAR PHYSICS

Fifth Series: The Statistical Theory of Nuclear Reactions Lecturer: V. F. Weisskopf

LECTURE XXXV. Resonance Processes

A) The Breit-Wigner Formula

The cross sections of nuclear reactions, which were calculated from the emission probabilities in Section XXXIII, were defined as average cross sections over a region of energy of the incoming particle, which includes many resonance levels. This section is devoted to the study of the cross section as a function of energy, if the energy is sharply defined. In this case we expect resonance phenomena as described qualitatively in Section XXXII.

We first investigate the energy relations: if ϵ_α is the energy of the incoming particle, the energy ϵ_β of an emitted particle is given by:

$$\epsilon_\beta = \epsilon_\alpha + B_a - B_b + E_\alpha - E_\beta \tag{38}$$

where E_α and E_β are the excitation energies of the initial and final nucleus respectively, and B_a and B_b are the binding energies of a to the initial nucleus or b to the final nucleus respectively. The energy ϵ_β is thus dependent on ϵ_α and is in general not equal to the energy ϵ_β^n of a particle b emitted by any of the excited states n of the compound nucleus, except if ϵ_α is just in the middle of a resonance. This is the case if $\epsilon_\alpha + B_a$ is just equal to the excitation energy of the compound state n.

The expression of the cross section of a nuclear reaction in the resonance region has been developed by Breit and Wigner[1] and later by Bethe and

1) Breit and Wigner, Phys. Rev. 49, 519 (1936)

Placzek[2], by Peierls and Kapur[3] and by Siegert[4]. In the following it is not attempted to give an exact derivation of it. The analogy to the resonance of damped oscillators is used, to make the main features of the formula as plausible as possible.

We first introduce the concept of effective widths. According to (29) the partial widths Γ_β^n depend on the energy of the outgoing particle by means of the factor $k_\beta^n T_b(\epsilon_\beta^n)$. Since the energy ϵ_β of the outgoing particle in a nuclear reaction is not necessarily equal to ϵ_β^n, it is useful to introduce the effective width $\gamma_\beta^n(\epsilon_\beta)$ which is a function of ϵ_β. It is obtained from the actual width by taking the energy dependent factor at the value ϵ_β instead of ϵ_β^n:

$$\gamma_\beta^n(\epsilon_\beta) \cong \Gamma_\beta^n \left(\epsilon_\beta/\epsilon_\beta^n\right)^{\frac{1}{2}} \frac{T_b(\epsilon_\beta)}{T_b(\epsilon_\beta^n)} \tag{42}$$

This relation is not quite exact because of the slight energy dependence of the other factors in (28), namely $\left|\Phi_\beta^n(R)\right|^2$; however, the resonance region does not extend further than the region of validity of (31), in which $\left|\Phi_\beta^n(R)\right|^2$ is not energy dependent.

Let us assume that the energy of an incoming particle a lies within an interval d which is much smaller than the width of a resonance level of the compound nucleus. Furthermore we assume that the energy ϵ is near to the energy ϵ^n at which the particle would be at the maximum of a resonance[5], and comparatively far away from other resonances so that we do not need to consider the influence of other levels. The cross section $\sigma(\epsilon)$ as a function of the energy ϵ has then the characteristic resonance dependence:

2) Bethe and Placzek, Phys. Rev. 51, 450 (1937).

3) Peierls and Kapur, Roy. Soc. Proc. 166, 277 (1938).

4) Siegert, Phys. Rev. 56, 750 (1939).

5) ϵ^n should, in accordance with Section XXXIV, be written ϵ_α^n, and ϵ should have an index ϵ_α. In this section we omit this index in the energy of the incident particle and ϵ^n the energy at which it is at resonance with the n[th] level.

$$\sigma(\epsilon) = F(\epsilon)\,\frac{\delta(\epsilon)}{(\epsilon-\epsilon^n)^2 + [\delta(\epsilon)/2]^2} \tag{43}$$

where $\delta(\epsilon)$ is the effective total width of the resonance, which is the sum of all effective partial widths:

$$\delta(\epsilon) = \sum_{\beta} \gamma_{\beta}(\epsilon_{\beta}) \tag{43a}$$

ϵ_{β} is the energy (given by (38)) with which the particle is emitted. The sum is taken over all possible index triples $b_{\beta} l'$. $F(\epsilon)$ is a slowly varying function of ϵ, which we determine later. If ϵ is in resonance, $\epsilon = \epsilon^n$, the energies ϵ_{β} are the same as they would be if the particles were emitted by the resonance level n independently. Thus $\delta(\epsilon^n) = \Delta^n$.

Expression (43) can be understood by analogy to the case of forced vibrations of a damped oscillator with a proper frequency ν under the influence of a periodic force F cos νt with a frequency ν. The damping is assumed to be due to a frequency dependent friction force $-m\delta(\nu)x$ (x is the displacement of the oscillator). One obtains for the displacement x:

$$x = \frac{F}{2m\nu_0}\left(\frac{1}{\nu_0-\nu+i\delta/2} + \frac{1}{\nu_0+\nu-i\delta/2}\right)e^{i\nu t} \tag{44a}$$

And for the average energy absorbed by the oscillator per unit time:

$$\frac{dE}{dt}\;\frac{1}{4m}\;\frac{F\delta(\nu)}{(\nu-\nu_0)^2 + [\delta(\nu)/2]^2} \tag{44b}$$

In this expression, terms smaller than of the order $(\nu_0-\nu)/(\nu_0+\nu)$ are neglected (second term in (44a)). The time dependence of the free vibration of this oscillator would be $x \sim \cos\nu_0 t\, e^{-\delta(\nu_0)t/2}$; its intensity in this case decays exponentially with the decay constant $\delta(\nu_0)$.

From this we can understand expression (43) for the cross section, by replacing in (44b) frequencies by energies. $\delta(\nu)$ goes over into the effective total width $\delta(\epsilon)$ of the level, whose value for $\epsilon = \epsilon^n$ is also (after division by $\hbar$) the actual decay constant of the level, just as $\delta(\nu_0)$ is the decay constant of the intensity of the oscillator. The function $F(\epsilon)$ in (43) is an analogue to a frequency dependent force F, acting on the oscillator.

We determine the function F( ) in (43) by comparing this cross section with its inverse: the decay of the compound nucleus with an emission of a particle with the energy ϵ within the small interval $d\epsilon$. The probability of this emission $P(\epsilon)d\epsilon$ must be proportional to the same resonance factor, which is contained in $\sigma(\epsilon)$, but should give, after integration over the total width of the level, the total emission probability Γ_α^n (the partial width for the emission of the particle a)

$$\Gamma_\alpha^n = \int_{\epsilon^n-\Delta\epsilon}^{\epsilon^n+\Delta\epsilon} P(\epsilon)\,d\epsilon$$

Δ is an energy interval large compared to the width. This is fulfilled by

$$P(\epsilon)\,d\epsilon = \gamma_\alpha^n(\epsilon)\frac{\delta^n(\epsilon)/2\pi}{(\epsilon-\epsilon^n)^2+(\delta^n/2)^2}\,d\epsilon \qquad (45)$$

since

$$\gamma_\alpha^n(\epsilon)\frac{\delta(\epsilon)/2\pi}{(\epsilon-\epsilon^n)^2+(\delta^n/2)^2}\,d\epsilon \cong \gamma_\alpha^n(\epsilon^n) = \Gamma_\alpha^n$$

This relation is right if $\Gamma_\alpha^n(\epsilon)$ and $\delta^n(\epsilon)$ are sufficiently slowly variable over the resonance region.

It also would be fulfilled by putting the actual partial width Γ_α^n instead of the effective partial width $\gamma_\alpha^n(\epsilon)$ as the first factor in (45). It is evident, however, from the derivation of the energy dependence of $\gamma_\alpha^n(\epsilon)$ in (42) that the emission of a particle with an energy ϵ is proportional to $\gamma_\alpha^n(\epsilon)$ rather than to Γ_α^n.

We now apply the expression of detailed balance (16) to calculate the function $F(\epsilon)$ in (43) from (45). Since the end state of the capture process has the weight unity in this case, $\omega_c(E)dE$ must be replaced in (16) by unity. We obtain instead of (16):

$$P(\epsilon)\,d\epsilon = \frac{\sigma(\epsilon)}{2\pi^2 \lambda\!\!\!^{-2}}\,d\epsilon$$

where $\lambda\!\!\!^-$ is the wave length corresponding to ϵ. From this we get by using (45):

$$\sigma(\epsilon) = \pi \lambda\!\!\!^{-2} \frac{\gamma_a^n(\epsilon)\,\delta^n(\epsilon)}{(\epsilon-\epsilon^n)^2 + [\delta^n(\epsilon)/2]^2} \tag{46}$$

This is the cross section for capture of the particle a regardless what reaction should follow. Since the δ^n in the numberator of (46) was written in (43a) as a sum of the partial widths belonging to all processes which can be initiated by the capture of a, it is self evident that the cross section for a special nuclear reaction $\alpha \rightarrow \beta$ is given by:

$$\sigma(\alpha,\beta) = \pi \lambda\!\!\!^{-2} \frac{\gamma_a^n(\epsilon)\,\gamma_b^n(\epsilon)}{(\epsilon-\epsilon^n)^2 + [\delta^n(\epsilon)/2]^2} \tag{47}$$

If the resonance energies ϵ^n of several levels are near enough to ϵ, the different contributions add up coherently: the amplitudes and not the intensities must be summed. One then obtains an expression, whose form again can be understood from the analogy with a forced oscillation. The amplitude according to (44a) is proportional to the complex factor $1/(\nu-\nu_0+i\gamma/2)$ which here appears in the form $1/(\epsilon-\epsilon^n+i\delta^n/2)$.

The cross section of an $(a\alpha l, b\beta l')$ reaction with the initial particle a incident with an energy ϵ and a given angular momentum $l\hbar$ upon the initial nucleus in the state α, resulting in an emission of b with an angular momentum l' and with the final nucleus in β, is given by:

$$\sigma(\alpha,\beta) = \pi \lambda\!\!\!^{-2} \left| \sum_n \frac{\gamma_{a\alpha l}\,\gamma^n_{b\beta l'}}{\epsilon-\epsilon_n+i\delta^n/2} e^{i\phi_{\alpha\beta}} + f(\epsilon) \right|^2 \tag{48}$$

Here ϵ_n are the resonance values: $\epsilon_n = E_n - B_a - E_\alpha$. E_n is the energy of the excited state n of the compound nucleus acting as intermediate state, E_α is the energy of the state α of the initial nucleus; $\phi_{\alpha\beta}$ is the phase difference of the initial and the final state and it vanishes if $\alpha=\beta$. $\gamma^n_{a\alpha l}$ and $\gamma^n_{b\beta l}$ are the effective partial widths of n, corresponding to the emission of a and b respectively, with the residual nucleus left in α or β. $f(\epsilon)$ is a slowly varying function of ϵ_a which is small compared to the resonance contributions of the first term. It is introduced to replace the contributions of the second term in the amplitude (44a) and to include the contributions from the continuous region far off the resonance which is not included in the first term.

The energy dependent parts of the widths are common to all the terms in the sum over intermediate states n and one can write according to (42)

$$\sigma(\alpha,\beta)=\frac{\pi\hbar^4}{m^2}\frac{k_b}{k_a}T(\epsilon_a)T(\epsilon_b)\left|\sum_n\frac{\Phi^n_{a\alpha\ell}\Phi^n_{b\beta\ell'}}{(\epsilon_a-\epsilon^n)+i\delta^n/2}+f'(\epsilon)\right|^2 \qquad (49)$$

k_a and k_b are the wave numbers corresponding to the energies ϵ_a and ϵ_b of the particles a or b respectively. Here the phases ϕ_α and ϕ_β have been incorporated in the Φ's. $f'(\epsilon)$ is the same as $f(\epsilon)$ after division by the factors taken out. The numerator of the resonance terms consists here only of nuclear magnitudes independent of the energies of the particles.

In formulas (48) and (49) all states α, β, n of the nuclei are considered as of weight unity. If some of these states are degenerated it is sometimes more useful to sum over all substates. The total cross section is the sum of all cross sections from every single substate of α to every one of β. In case of angular momentum degeneracy these summations can be performed and Bethe and Placzek have derived the following expressions:

$$\sigma(aA,bB)=\frac{\pi\lambda^2}{(2s+1)(2i+1)}\sum_{\ell j\ell' j'}\sum_J(2J+1)\left|\sum_0^J\frac{\Phi_{a\ell jA}\Phi_{b\ell' j'B}}{\epsilon_a-\epsilon_c+\frac{1}{2}i\delta^c}+f(E)\right|^2 \qquad (50)$$

Here s is the spin of the incident particle a, i is the spin of the initial state A. The initial and final states are no denoted with capital letters (A,B) instead of the greek letters (α,β) in order to indicate that they comprise all magnetic substates. The compound state is called C. ℓ,ℓ' and j,j' are the orbital and the total angular momentum of the incident or emitted particle respectively; J is the total angular momentum of incoming particle and initial nucleus and therefore also the total spin of the compound state C.

$\Phi^C_{a\ell JA}$ is the amplitude Φ of the compound state C in respect to the emission of the particle a, with orbital and total angular momentum ℓ and j and with the residual nucleus in A. It is evident that $\Phi^C_{a\ell JA}=0$ unless $|j+i|>j>|j-i|$.

The summation sign $\sum^J$ indicates that the sums should be taken only over states C with the spin J. The summation over J is outside of the absolute square signs which means that the contributions to the scattering with different total angular momentum do not interfere.

In most cases in which (50) is applied, the contribution of one compound level only is essential. The others are far off resonance. We then obtain

$$\sigma(aA,bB) = \pi \lambda\!\!\!^{-}{}_a^2 \frac{2J+1}{(2s+1)(2i+1)} \frac{\gamma_{aA}^C \gamma_{bB}^C}{(\epsilon_a-\epsilon_c)^2+(\delta^c/2)^2} \qquad (51)$$

Here $\gamma_{aA}^C = \sum_{\ell J} \gamma_{a\ell A}^C$ is the partial width of C for the emission of a with a residual state A irrespective of the spin and of the momentum of the emitted particle[6]. All substates have, of course, the same partial width, because of the spatial symmetry.

The formulas can be extended to include also radiative phenomena such as the emission of light quanta by the nucleus instead of a particle b in a nuclear reaction. The cross section for a nuclear reaction in which a particle a is incident upon a nucleus in the state α and a light quantum is emitted which leaves the compound nucleus in the state ρ is given by:

$$\sigma(a\alpha,r\rho) = \pi \lambda\!\!\!^{-}{}_a^2 \left| \sum_n \frac{(\gamma_{a\alpha}^n \gamma_{r\rho}^n)^{\frac{1}{2}} e^{i\phi_{a\rho}}}{(\epsilon_a-\epsilon^n)-i\delta^n/2} + f(\epsilon) \right|^2 \qquad (52)$$

$\gamma_{r\rho}^n$ is the effective partial width for radiation. It differs from the actual partial radiation width $\Gamma_{r\rho}^n$ by a factor $(\nu_\rho/\nu_\rho^n)^{2\ell+1}$ where ν_ρ^n is the frequency which would be emitted if the state n actually went over to the state ρ by a radiative transition, and where ν_ρ is the frequency which is emitted in the radiative capture ending in the state ρ. ℓ in the exponent depends on the multipole character of the emitted radiation; it is 1, 2... for dipole or quadrupole (etc) radiation.

If only one compound level is important, the total cross section for radiative capture irrespective of the level ρ is given by

$$\sigma(aA,r) = \pi \lambda\!\!\!^{-}{}_a^2 \frac{2J+1}{(2s+1)(2i+1)} \frac{\gamma_{aA}^C \gamma_r^C}{(\epsilon_a-\epsilon^n)^2+(\delta^C/2)^2} \qquad (53)$$

---

6) In general, if C and A have angular momenta different from zero, different values of ℓ are possible. Assuming, for example, $J = 3/2$, $i = 1$, a particle with spin $\frac{1}{2}$ can be emitted with $\ell = 0,1,2,3$.

$\gamma_r^c = \sum_\rho \gamma_{r\rho}^c$ is the total effective radiation width.

B) Discussion of Resonance Phenomena

A few general conclusions can be drawn from the one-level formula (47).

If there is only one emission process possible from the compound nucleus, which then necessarily is the reemission of the absorbed particle, we can put $\delta^n = \gamma_\alpha^n$ and we get for the only possible process, the elastic scattering of a:

$$\sigma(\alpha,\alpha) = \pi \lambda\!\!\!^{-2} \frac{(\gamma_\alpha^n)^2}{(\epsilon-\epsilon^n)^2 + [\gamma_\alpha^n(\epsilon)/2]^2} \tag{54}$$

The value at resonance is $4\pi\lambda\!\!\!^{-2}$, and this is the maximum possible value for elastic scattering.

If there is one competing process, say the emission of b, we obtain $\delta^n = \gamma_\alpha^n + \gamma_\beta^n$ and the elastic scattering is then given by

$$\sigma(\alpha,\alpha) = \pi \lambda\!\!\!^{-2} \frac{(\gamma_\alpha^n)^2}{(\epsilon-\epsilon^n)^2 + (\gamma_\alpha^n + \gamma_\beta^n)^2/4}$$

which is off resonance, $(\epsilon - \epsilon^n) \gg \Delta^n$ the same as (54) but, in resonance, weaker than (54). The existence of a second emission possibility diminishes the elastic cross section only near resonance. The cross section $\sigma(\alpha,\beta)$ for the emission of b is given by (47), and it is easily seen that the maximum cross section possible in resonance in this case is $\pi\lambda\!\!\!^{-2}$ which is assumed if $\gamma_\alpha = \gamma_\beta$.

We now apply the resonance expressions to the case of slow neutron reactions. There we assume that the angular momentum of the incoming particle is zero. There are four reactions possible with slow neutrons: 1) elastic reemission of the neutron; the energy is not sufficient to leave the nucleus excited by an inelastic collision. 2) Emission of a γ-ray, in which case the neutron is permanently captured, since, after emission, the excitation energy of the compound nucleus falls below the binding energy of the neutron. 3) Emission of an α-particle. 4) Emission of a large fragment; with very heavy nuclei the fission of the nucleus can be induced by slow neutrons.

The emission of a proton is impossible as can be seen by the following consideration: if the proton would be emitted with higher energy than the one of the neutron, the energy of the residual nucleus would be lower than the one of the initial nucleus, and the two nuclei would be isobars differing by one unit of charge. Thus the initial nucleus would be β-unstable, which is impossible. If the emitted proton had equal or less energy than the neutron it would not be able to penetrate the potential barrier.

We first assume that there is only one level sufficiently near to the energy to which the compound nucleus can be excited by the neutron. We then apply (51) and use (33) for the partial width γ_{aA}^{C} of the neutron. Furthermore, $J = i \pm \frac{1}{2}$ since the neutron can contribute only its intrinsic spin, $s = \frac{1}{2}$, and we get

$$\sigma(a,b) = \pi \lambda\!\!\!^{-2} \frac{1}{2}\left(1 \pm \frac{1}{2i+1}\right) \frac{C\sqrt{\epsilon}\,\gamma_{bB}^{C}}{(\epsilon - \epsilon_C)^2 + (\delta^C/2)^2}$$

The cross section for the elastic scattering of slow neutrons is then:

$$\sigma_{e\ell} = \pi \frac{\hbar^2}{2m} \frac{1}{2}\left(1 \pm \frac{1}{2i+1}\right) \frac{C^2}{(\epsilon - \epsilon_C)^2 + (\delta^C/2)^2} \qquad (55)$$

which is obtained by putting $\gamma_{GB}^{C} = C\sqrt{\epsilon}$ also according to (33).

The cross section for emission of a light quantum or another particle (both are denoted by b) is given by

$$\sigma(a,b) = \frac{\pi\hbar^2}{2m\epsilon^{\frac{1}{2}}}\left(1 \pm \frac{1}{2i+1}\right) \frac{C\gamma_{bB}^{C}}{(\epsilon - \epsilon_C)^2 + (\delta^C/2)^2}$$

This cross section shows the characteristic $\epsilon^{-\frac{1}{2}}$ dependence on the neutron energy (the so-called 1/v law). If the width δ^C is very large, the $\epsilon^{-\frac{1}{2}}$-factor is the determining factor for the energy dependence. This is the case e.g., in the boron reaction:

$$n + B^{10} = Li^7 + \alpha$$

where the total width is determined mainly by the α-emission width. The α-particles are emitted with an energy of about 2.7 Mev, and are able to pass above the barrier. The corresponding width γ_{α}^{C} is therefore very large and was found to be about 200 Kev. Thus the boron reaction follows the 1/v law up to energies ϵ small compared to 200 Kev.

If the only reaction possible after capture of a neutron is the emission of a light quantum, (as in most cases of heavy nuclei) the capture cross section for a slow neutron is given by:

$$\sigma_{capture} = \frac{\pi\hbar^2}{2m\epsilon}\frac{1}{2}\left(1 \pm \frac{1}{2i+1}\right)\frac{C\Gamma_r^C}{(\epsilon-\epsilon_C)^2+\frac{1}{4}(C\sqrt{\epsilon}+\Gamma_r^C)^2} \qquad (56)$$

This expression is right if only one resonance level determines the process. It must be generalized by means of (52) for the case of several resonances near ϵ .

C) Experimental Material

A number of resonance phenomena have been observed with bombardments of light nuclei. Staub and Tatel[7)] have observed a strong resonance in the elastic scattering of neutrons by helium. Thus resonance could be assigned to a doublet level, excited by p-neutrons and was analysed by means of a formula similar to (54).

Herb and collaborators in Wisconsin[8)] and Tuve et al[9)] have found resonances by measuring the emitted γ-ray intensity as a function of the energy of the bombarded protons on Li, F, Al. The γ-rays are either emitted by the compound nucleus itself or by the excited final nucleus which is left over after emission of an α-particle (the latter possible occurs in F). The observed γ-ray intensity shows sharp maxima if the energy of the bombarding protons is such that a level is excited in the compound nucleus. Thus the resonances give a picture of the spectrum of the compound nucleus above the proton binding energy. In Li and F only those levels are found which, because of selection rules, cannot emit an α-particle. The levels that emit α-particles are so broad that no resonance can be observed. In Al, however, the α-emission is made less probable by the higher barrier, and therefore, also α-emitting levels show

7) Staub and Tatel, Phys. Rev. 58, 820 (1940).

8) Herb, Kerst, McKibben, Phys. Rev. 51, 691 (1937).
Bernet, Herb, Parkinson, Phys. Rev. 54, 398 (1938).
Plain, Herb, Hudson, Warren, Phys. Rev. 57, 187 (1940).

9) Hafstad, Heydenburg, Tuve, Phys. Rev. 50, 504 (1936).

resonance. The most striking result of a comparison between the three elements is the greater level density in the heavier nuclei. The average level distance is about 500 Kev in Li, 200 Kev in F, and 30 Kev in Al. The observed widths of the levels are a sum of particle and radiation widths. They are 11 Kev in Li, about 8 Kev in F and of the order of one or two Kev in Al.

The level density in Al is so high that one would not expect to be able to find resolvable resonances in elements much heavier than Al with charged particle beams. Resonances in heavier nuclei were found with slow neutrons only. The formula (56) for the capture of slow neutrons can then be tested, and the different magnitudes: ϵ_C ; δ^C and C, can be determined. In this case, the neutron width $C\sqrt{\epsilon}$ is usually much smaller than the radiation width. We therefore put $\delta^C \approx \gamma_r^C$ and the formula (56) reduces then to:

$$\sigma_{capture}\ \sigma_0 \frac{(\epsilon_C/\epsilon)^{\frac{1}{2}}}{1+\left[(\epsilon-\epsilon_C)/(\delta^C/2)\right]^2} \qquad (56a)$$

Where σ_0 is the cross section at resonance:

$$\sigma_0 = 2\pi\lambda_C^2\left(1\pm\frac{1}{2i+1}\right)\frac{C\sqrt{\epsilon_C}}{\Delta^C} = \frac{1.33}{\epsilon_C}\left(1\pm\frac{1}{2i+1}\right)\frac{C\sqrt{\epsilon_C}}{\Delta^C}\times 10^{-18}\,cm^2 \qquad (56b)$$

In the second expression the energy ϵ_C is expressed in ev.

In this formula the Doppler effect has been neglected. Because of the thermal motion of the nuclei, the relative energy between neutron and nucleus is not exactly equal to the neutron energy. The relative energy ϵ is distributed around the neutron energy E with a Gaussian distribution: $1/\sqrt{2\pi K}\exp\left[-(E-\epsilon)^2/2K^2\right]$ where the Doppler width K is given by

$$K = 2(m\epsilon kT/M)^{\frac{1}{2}}$$

and m/M is the ratio of the masses of the neutron and the nucleus.

Since (56a) gives the dependence of $\sigma(\epsilon)$ on the relative energy, the cross section as a function of the actual energy E is given by:

$$\sigma(E) = \frac{1}{\sqrt{2\pi K}}\int_{-\infty}^{+\infty}\sigma(\epsilon)\exp\left[-(E-\epsilon)^2/2K^2\right]d\epsilon \qquad (57)$$

If the Doppler width K is small compared to the natural line width δ^C, the Doppler effect is negligible and (56a) can be used with ϵ as the actual neutron energy. If K is large compared to δ^C, the shape of the absorption line near its maximum is given entirely by the Doppler effect and we get:

$$\sigma(\epsilon) \approx \frac{\pi \lambda\!\!\!^{-2}}{\sqrt{2\pi}K} \frac{C\sqrt{\epsilon}}{K} \left(1 \pm \frac{1}{2i+1}\right) e^{-(\epsilon-\epsilon_C)^2/2K^2} \qquad (56c)$$

However if the energy is sufficiently far away from resonance, $\epsilon - \epsilon \gg K$ σ is again given by (56a), since (56c) falls off exponentially, whereas (56a) falls off only quantratically with $(\epsilon - \epsilon_C)$.

The following methods are used to determine the characteristic magnitudes ϵ_C, C and Δ^C of a resonance level:

1) The direct method of measuring the absorption of, or the radioactivity produced by, a monochromatic neutron beam with variable energy. Experiments with monoenergetic neutrons have been first performed by Alvarez by means of a modulated neutron source and by a suitably modulated amplifier, which registers only neutrons which have arrived at the counter in a specified time interval after their production. Bacher and Baker have improved this method and were able to extend the range of energies investigated from zero to about 5 ev.

This method gives $\sigma_{capture}$ directly as a function of the energy ϵ of the neutrons. The quantity measured in an absorption measurement is actually the sum of $\sigma_{capture}$ and the scattering cross section. For low energies and near resonance, the scattering cross section can be neglected compared to $\sigma_{capture}$.

2) The boron absorption method. The boron absorption method is used for the determination of the energy of the resonance levels. Advantage is taken of the 1/v-dependence of the cross section of the $(n\alpha)$ process in boron, which is responsible for the absorption of neutrons in boron. The principle of this method consists of the following: a material is activated by absorption of neutrons from a beam containing a continuous spectrum of neutron energies. The reduction of the activation is measured after covering the material with boron. This reduction is proportional to the absorption in boron and therefore proportional to $\epsilon_C^{-\frac{1}{2}}$ if ϵ_C is the energy of the neutrons at resonance. As shown before, every material captures to a certain extent very slow neutrons of thermal

energy. The thermal neutrons can be eliminated by covering the material with cadmium. The value of ϵ_C can be obtained by measuring the reduction of the activity by boron of the cadmium covered material in which only the resonance neutrons are measured and by comparing this with the difference in reduction if the cadmium is removed. The latter difference is due to the absorption of thermal neutrons in boron. We then obtain $\epsilon_C/\epsilon_{th} = (k_{res}/k_{th})^2$ where k_{res} and k_{th} are the absorption coefficients of the resonance neutrons and the thermal neutrons respectively. ϵ_{th} is a suitably chosen average energy of the thermal neutrons which is, for boron absorbers, given by $\epsilon_{th} = (\pi/4)\,kt$

The boron method can also be used to ascertain that essentially one level formula may safely be applied. If this is true, the reduction of the activity in the cadmium covered material should depend on the thickness d of the boron absorber according to an exponential function e^{-kd}. If two or more levels contribute with similar intensity, the dependence on d would be a sum of exponentials. Results on silver, for example, show a behavior of that nature.

3) The self absorption method. The absorption of a neutron beam with a continuous spectrum over the line width is measured as a function of thickness d of one absorber, with a radioactive indicator of the same material. The thermal neutrons are removed by cadmium shielding. The measured absorption starts with an exponential dependence on d and goes over into $a\sqrt{d}$ dependence when the inner parts of the line are no longer contributing to an additional absorption, which happens when all the intensity in the middle of the time is absorbed. The evaluation of this dependence gives a means to determine σ_0 the cross section of capture at the maximum of resonance, if there is only one level which mainly contributed.

Let us assume the intensity I_0 of the neutron beam is measured by the indicator without any absorber. I_0 is reduced to I by an absorber of the same material as the indicator and of a thickness d. For thin absorbers the ratio I/I_0 is given by $I/I_0 = \exp(-n\sigma_s d)$ where n is the number of atoms per cm^3 and

$$\sigma_s = \frac{\int \sigma_c^2(\epsilon)\,d\epsilon}{\int \sigma_c(\epsilon)\,d\epsilon} = \frac{1}{2}\,\frac{\sigma_0}{W} \text{ for } n\sigma_s d \ll 1$$

Here σ is defined in (56b) and W is a function of the ratio $\eta = \Delta^c/K$ of the natural width and the Doppler width:

$$W = 1 + (1/\eta^2) + \ldots\ldots\ldots \text{ for } \eta \gg 1$$

$$W = \sqrt{2/\pi}\,(1/\eta) + 1 + \ldots\ldots \text{ for } \eta \ll 1$$

W is unity for $\delta^c \gg K$.

If the absorption in a very thick layer is measured by an indicator of the same material, the middle of the resonance is completely absorbed and only the wings of the line are measured. In the wings the Doppler effect is no longer important and the intensity is given by:

$$I/I_0 = 1/\sqrt{\pi\sigma_0 nd}\,, \text{ for } n\sigma_0 d \gg W \qquad (58)$$

Relation (58) is correct if $\sigma_0 nd \gg W$, which means that the energy regions affected by the Doppler effect are completely absorbed.

4) The activity method. The activity produced by a neutron beam of given spectrum is measured, after shielding off the thermal neutrons. The neutron spectrum is usually obtained by slowing down high energy neutrons in light material as carbon or paraffin. The intensity per energy interval $d\epsilon$ is then proportional to $d\epsilon/\epsilon$.

This method gives a measure of

$$\int \frac{\sigma(\epsilon)}{\epsilon}\,d\epsilon = 2\pi\sigma_0 \frac{\Delta^c}{\epsilon_c}$$

independent of the Doppler effect.

5) The measurement of the thermal absorption cross section. The thermal absorption can be used for the determination of level constants only if the energy of all higher levels is large compared to the lowest one, and if no level below zero energy is of any influence. The lowest level must be low enough so that the absorption at thermal energies is taking place in the "wing" of the resonance time.

In case the one-level formula is applicable, we obtain:

from method 1): ϵ_C, Δ^C, C

from method 2): ϵ_C .

from method 3): σ_0 if $\Delta^C >> K$ or from thick layers,

$\sigma_0 \Delta^C$ if $\delta^C << K$

from method 4): σ_0/ϵ_C

from method 5): $\sigma_{th} = \pi \lambda_{th}^2 \frac{C\sqrt{\epsilon_{th}}}{\epsilon_C^2}$

Here λ_{th} and ϵ_{th} are the wave lengths and the energies for thermal energies. Thus the three unknown quantities ϵ_C, C , Δ^C can be determined. Inconsistencies within the five methods can be due to the influence of other levels. σ_{th} can be unduly small because of the existence of a negative level, or higher because of the contribution of other levels. Methods 3) and 4) may give too large values because they include the effect of higher levels.

A table of the characteristic values for some levels of different elements is given below. Only elements are included where the evidence seems to be not too contradictory. The numbers in parenthesis at the references indicate the methods used, as listed previously.

In Table I, $C' = \frac{1}{2}\left(1 \pm \frac{}{2I+1}\right)C$. The plus or minus sign refers to the cases where the spin of the neutron is added to, or subtracted from, the spin of the initial nucleus. The value of Δ^C in As cannot be determined because the Doppler width in this case is dominant.

The radiation width, Γ_r which, in these cases, is practically equal to the total width Δ^C , is very nearly equal in all cases. It shows unusually little fluctuation. This relative constancy is connected with the fact that represents a sum of many partial widths corresponding to radiative transitions to a great number of lower states. The fluctuations of single transition probabilities are averaged out. The values of C' show somewhat greater fluctuations. They represent a single transition probability and may depend more critically on the properties of the individual level.

The values C' seem to fulfill the condition (36) fairly well. The value

$C\sqrt{\epsilon}$ for $\epsilon \sim 1$ Mev is between 0.2 to 0.7 ev. The resonance level density at that energy ought to be larger than $2\pi C\sqrt{\epsilon}$ which lies between 1.2 and 4.5 ev for the elements in Table I. From all that we know, it is of this order of magnitude (see XXXVI) and this proves that C is near to the largest value which it is allowed to assume.

TABLE I

Bombarded Element	Life time of product	ϵ_C(ev)	Ref.	Γ_r	Ref.	ϵ_nx10^4	C'x10^4	Ref.
As^{76}	27 h	40	$1^{(2,3)}$			32	5	$1^{(3)}$
Rh^{104}	44 sec	0.84	$4^{(2)}$	0.15 0.13	$6^{(3,4)}$ $3^{(3,4)}$	1.8 1.8	2 2	$3^{(3)}$ $2^{(3)}$
Cd^{117}	3.75 h	0.14	$5^{(1)}$	0.12	$5^{(1)}$	2.5	6.6	$5^{(1)}$
In	54 min	1.0	$5^{(1)}$	0.07 0.07	$5^{(1)}$ $3^{(3,4)}$	55	5	$3^{(3)}$
Ir^{192}	19.5 h	~1.0	$2^{(2)}$	~0.1	$2^{(3,4)}$	~2	~2	$2^{(3)}$
Au^{198}	2.7 days	3.5 2.6	$1^{(2)}$ $2^{(2)}$	0.1	$6^{(3,4)}$	6	3.2	$6^{(3)}$

1) O. Frisch Det Kg 2 Danske Vid. Selskob Medd. XIV 12 (1937)
2) Jaeckel Phys. 107, 669 (1937)
3) Hornborstel, Goldsmith, Mayley Phys. Rev. 58, 18 (1940)
4) Horveth, Salant Phys. Rev. 59, 154 (1941)
5) Baker, Bacher Phys. Rev. 59, 332 (1941)
6) Feeny, Lapointe, Rasetti Phys. Rev. 61, 469 (1942)

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LECTURE SERIES ON NUCLEAR PHYSICS

Fifth Series: The Statistical Theory of Nuclear Reactions

Lecturer: V. F. Weisskopf

LECTURE XXXVI. Non Resonance Processes

A) General Discussion

If the energy of the incident particle is higher than a certain limit, resonances do no longer occur. This can be due to experimental reasons: the energy cannot be made monochromatic enough to resolve the levels; it can be due to the Doppler effect; the Doppler width reaches, for example, 50 ev for an incident particle energy of 250,000 ev, if the nucleus is 100 times heavier than the particle; 50 ev is of the order of or larger than the average distance D between resonances for heavy nuclei. Furthermore, at energies of several Mev the actual width of the level reaches the order of level distances. This can be seen in the following way: according to (17) the partial width for the emission of a neutron after leaving the nucleus in a given state, reaches the value $(2\ell+1)D/2\pi$. If the residual nucleus can be left in several excited states, the corresponding widths add up and give rise to a total width, equal or larger than the level distance D. Thus, even under ideal conditions of a monochromatic beam and zero temperature, resonances disappear at an energy of several Mev's.

In order to compute the cross sections of non-resonance processes, we use Bohr's assumption B (see introduction). The cross section can then be expressed by (2)

$$\sigma(a,b) = \sigma_a \eta_b \tag{2}$$

where σ_a is the cross section of the formation of the compound nucleus, and η_b

is the relative probability of the emission of the particle b:

$$\eta_b = \bar{\bar{\Gamma}}_b / \bar{\Delta}$$

Here $\bar{\bar{\Gamma}}_b$ is the partial width for the emission of b irrespective of in what state the residual nucleus is left: $\bar{\bar{\Gamma}}_b = \sum_{\beta l} \bar{\bar{\Gamma}}_{b\beta l}$ averaged over the levels of the compound states excited by a, $\bar{\Delta}$ is the averaged total width.

The partial widths $\bar{\bar{\Gamma}}_\alpha$ (we use again the abbreviated index α; see page 7, part XXXIII) are connected with the cross section σ_α of the inverse process: the formation of a compound nucleus by the particle a of angular momentum l and a nucleus in the state α. This relation is given by the expression (16):

$$\bar{\bar{\Gamma}}_\alpha = \frac{\sigma_\alpha}{2\pi^2 \lambda\!\!\!^{-2}} \frac{1}{\omega_c(E)} \tag{16a}$$

The cross section σ_a appearing in (2) is $\sum_l \sigma_{a\alpha l}$ with α being the ground state of the initial nucleus. Thus we get for (2):

$$\sigma(a,b) = \sum_l \sum_{l'} \sum_\beta 2\pi^2 \lambda\!\!\!^{-2} \omega_c(E) \left(\bar{\bar{\Gamma}}_\alpha \bar{\bar{\Gamma}}_\beta / \bar{\Delta}\right) \tag{2a}$$

where the summation over l and l' refers to the angular momentum index in the abbreviated index α or β respectively.

The consistency of the expressions (2), (3), (16), with the resonance formula (48) is shown by averaging (48) over an energy region $\Delta\epsilon$, which contains many resonances. If the level distances are large compared to the widths, this integration is a sum over the contributions of the energy regions around every single level. One then obtains:

$$\bar{\sigma}(\alpha,\beta) = \frac{2\pi^2 \lambda\!\!\!^{-2}}{\Delta\epsilon} \sum_n \frac{\Gamma_\alpha^n \Gamma_\beta^n}{\Delta^n}$$

The sum is extended over all levels within the energy interval $\Delta\epsilon$. This relation is right if the term $f(\epsilon)$ in (48) is sufficiently small compared to the resonance contributions. After introducing the average width $\bar{\Delta}$ over neighboring levels, we obtain:

$$\sigma(\alpha,\beta) = \frac{2\pi^2 \lambda\!\!\!^{-2}}{D} \frac{\bar{\bar{\Gamma}}_\alpha \bar{\bar{\Gamma}}_\beta}{\bar{\Delta}}$$

where $D = [\omega_c(E)]^{-1}$ is the average level distance. After summing over l and l'

and β we obtain the same expression as (2a). This shows that expression (2) for a non-resonance reaction is in agreement with the resonance formula in the region in which the latter one is valid ($D \gg \overline{\Delta}$).

We now proceed to the quantitative discussion of the magnitudes occurring in (2). We make use of the intimate connection between partial widths $\overline{\Gamma}_\alpha$ and cross sections which is given in (16a). In some energy regions the cross sections are better known, in others the widths; either magnitude can be used to determine the other. We distinguish three energy regions for the emitted (or absorbed) particle. The characteristic magnitude to distinguish the three regions is K_a which is the wave number of the particle near the nuclear surface and which is defined as follows:

$$K_a = i\left[(\partial\psi/\partial r_a)/\psi\right]_{r_a=R}$$

K_a is equal to k_a if there is no barrier. It is imaginary if the energy of the particle is insufficient to pass over the barrier. We compare K_a with k_o which is the wave number of the momentum p_o, $k_o = p_o/\hbar$, defined in (31). k_o is a wave number of the order d, where d is the distance between the nuclear constituents. The regions are defined as follows:

I. High energies: $K_a^2 > k_o^2$.

II. Low energies: $|K_a|^2 < k_o^2$. This region includes all energies near to the one for which it just passes over the barrier or, if there is no barrier, all energies smaller than $\hbar^2 k_o^2/2m$. It includes also energies which are not sufficient to pass over the barrier. In this case K_a^2 is negative.

III. Penetration region: $K_a^2 < -k_o^2$. This includes the cases of penetration of barriers with energies much lower than the barrier height.

In the first region, classical considerations are justified since the wave length of the particles is small compared to the nucleus and of the order or smaller of the distance between constituents. We may therefore assume that every particle that comes to the surface is absorbed. Hence we put $\sigma_{a\alpha,\ell}$ equal or nearly equal to its maximum value and get:

$$\left.\begin{aligned}\sigma_{a\alpha\ell} &= \xi(2\ell+1)\,\pi\lambda\!\!\!^{-}{}_a^{\,2}\\ \bar{\Gamma}_{a\alpha\ell} &= \xi(D/2\pi)(2\ell+1)\end{aligned}\right\}\text{Region I} \qquad (59a)$$

The second expression follows from (16a). ξ is a pure number smaller than unity which goes to unity for high energies and is introduced to take care of the transition regions.

The second energy region is the same region in which condition (31) in Section XXXIV is valid. The value of $\bar{\Gamma}_a$ can therefore be obtained from the expression (28) in a simple way. It was shown there that (28) can be written in the form (32):

$$\bar{\Gamma}_{\alpha} = C(E)\, k_a T_a(\epsilon_a) \qquad (32)$$

where C(E) is a function of the excitation energy E of the compound nucleus, which is independent of the nature (and angular momentum) of the emitted particle. In case of an s-neutron ($\ell = 0$, T = 1) this expression should join (59a) at $k_a = k_o$ and it is therefore tempting to put

$$C(E) = \xi(D/2\pi)(2\ell+1)(1/k_o)$$

where ξ may be some function of E only. Since, in the extent of the Region II, E does not vary by much, we may assume that the value of ξ is approximately constant in this region and of the order unity as in Region I. We then get:

$$\left.\begin{aligned}\Gamma_{a\alpha\ell} &= \xi\frac{D}{2\pi}\frac{k_a}{k_o}T_a(\epsilon_a)(2\ell+1)\\ \sigma_{a\alpha\ell} &= \xi\frac{k_a}{k_o}(2\ell+1)\pi\lambda\!\!\!^{-}{}_a^{\,2}\,T_a(\epsilon_a)\end{aligned}\right\}\text{Region II} \qquad (59b)$$

The second equation comes from (16a).

In the third region (high barrier), the wave length $\lambda\!\!\!^{-}{}_a$ will be always so small that the potential energy outside the nucleus does not change appreciably over a distance $\lambda\!\!\!^{-}{}_a$. Thus the WKB calculation is applicable and it yields

$$\left.\begin{aligned}\sigma_{a\alpha\ell} &= \xi(2\ell+1)\pi\lambda\!\!\!^{-}{}^{2}\,P_a\\ \Gamma_{a\alpha\ell} &= \xi P_a(2\ell+1)(D/2\pi)\end{aligned}\right\}\text{Region III} \qquad (60)$$

Here P_a is the penetration of the barrier, which is, in WKB approximation, given by

$$P = \exp\left[-\frac{2\sqrt{2m}}{\hbar}\int_{r_1}^{r_2}(V-E)^{\frac{1}{2}}\,dr\right]$$

Here r_1 and r_2 are the radii between which the potential energy V is larger than the total energy E. By comparison of this with the WKB expression (29) for T_a we obtain $P_a = (k_a/Ka)\ T_a$. Thus the regions II and III join smoothly.

It is interesting to investigate the Region II by means of the WKB method of approximation. This region is defined by the condition that the wave length $\lambda\!\!\!^{-}$ of the particle outside of the nucleus is long compared to the distance between the nuclear constituents. If we assume that $\lambda\!\!\!^{-}$ is still small enough to assume the potential fields outside the nuclear surface (centrifugal or Coulomb force) as slowly varying, it is allowed to apply the WKB approximation for the wave function from the outside up to the nuclear surface. There however, conditions change abruptly in distances small compared to $\lambda\!\!\!^{-}$. It seems to be appropriate to assume that, once a particle has penetrated this surface, it will not return without change in energy; in other words, it will have formed a compound nucleus. Thus the penetration process is equivalent to one dimensional problem with a potential energy of the type indicated in Fig. 2. Here a particle comes from the left side over a smooth potential barrier. At $x = 0$, the potential abruptly drops to a value which would give to the particle a wave length $\lambda\!\!\!^{-}$ comparable to the wave length inside of the nucleus, which is of the order of the

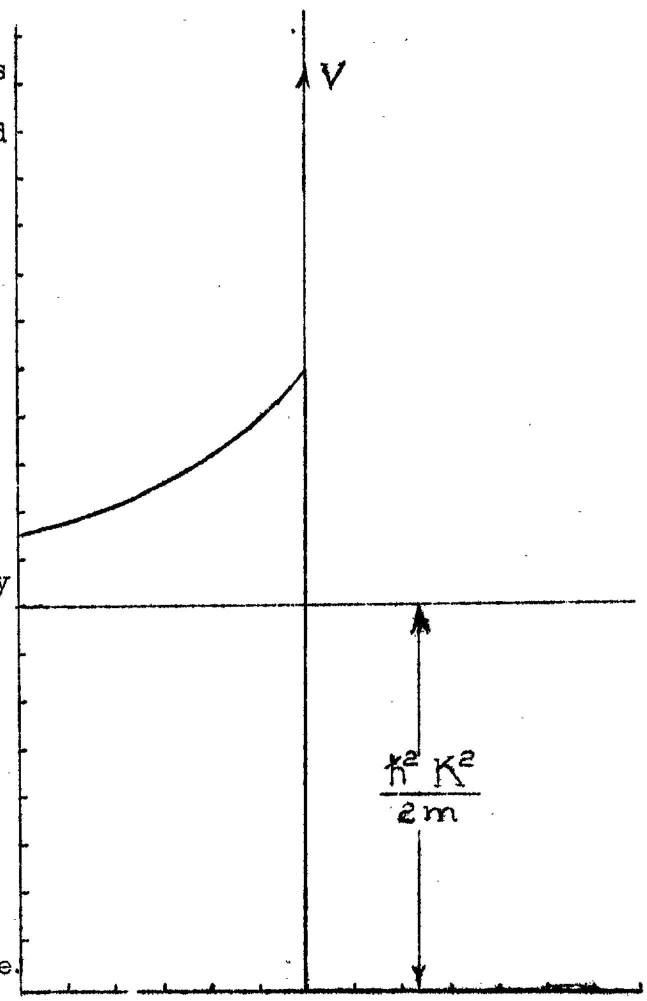


Fig. 2

distance between nuclear particles. (We call the corresponding wave number: K.) This region extends to the infinite, so that the particle does not return. If a wave of particles with an energy E (A particles per second) comes from the left, it is partially reflected (even if there is no barrier, because the potential energy changes abruptly at $x = 0$). The current which is not reflected but penetrates to the right (B particles per second) can be calculated easily by WKB method and gives

$$B = 4A\frac{\phi}{K}\left[\frac{1}{1+\phi/K}\right]^2 e^{-2C}$$

with

$$C = \frac{\sqrt{2m}}{\hbar}\int_{x_1}^{0}(V-E)^{\frac{1}{2}}dx, \quad \phi = \frac{\sqrt{2m}}{\hbar}(V_0-E)^{\frac{1}{2}}$$

where x_1 is the point at which the potential energy V becomes larger than the total energy E of the particle, and V_0 is the value of V at $x=0$. B/A is the ratio of the penetrating particles to the incident ones. According to this expression, the cross section for the formation of the compound nucleus is given by the maximum possible cross section, multiplied with B/A:

$$\sigma_\alpha = \pi \lambda\!\!\!^{-2}(2l+1)\cdot 4(\phi/K)e^{-2C} \qquad (61)$$

since $\phi \ll K$. By using the expression (28) for the WKB form of T, we can write:

$$T_a = (\phi/k_a)e^{-2C}$$

Thus, expression (61) is identical with the formulas (59) if we put

$$k_0 = \tfrac{1}{4}K \qquad (62)$$
$$\zeta = 1$$

Hence the limiting wave number k_0 for the three groups will be somewhat smaller than the reciprocal value of the distance between nuclear particles*). The energy $\hbar^2K^2/2m$ corresponding to the wave number K is of the order of 20 Mev, since the distance between the nuclear constituents is of the order 10^{-13}cm. The energy ϵ_0 corresponding to k_0 is then 16 times smaller, and of the order of 1 or 2 Mev.

Thus the three energy regions of the incoming particle are defined as follows:

*) These considerations are similar to the ones put forward in a paper by H. A. Bethe, Phys. Rev. 57, 1125 (1940).

Region I: The energy is more than 1 or 2 Mev above the barrier.

Region II: The energy is less than 1 or 2 Mev above or below the barrier.

Region III: The energy is more than 1 or 2 Mev below the barrier.

We now apply these considerations to the calculation of neutron cross sections. The total cross section σ_n for the formation of a compound nucleus by neutrons, is given by $\sigma_n = \sum_\ell \sigma_\ell$ where σ_ℓ is the contribution from the angular momentum $\hbar\ell$. We first consider neutrons of low energy $\epsilon \ll \epsilon_0$ where ϵ_0 is the energy $\hbar^2 k_0^2/2m$. For values of ℓ for which ϵ is larger or of the order of the centrifugal potential $\hbar^2 \ell(\ell+1)/2m R^2$, σ_ℓ should be calculated according to the rules (59b) valid in Region II:

$$\sigma_\ell = \xi\sqrt{\epsilon/\epsilon_0}\ (2\ell+1)\pi \lambda\!\!\!^{-2}\, T_{\ell a}(\epsilon)$$

$T_{\ell a}(\epsilon)$ for neutrons is given in (27d). For values of ℓ, for which $\hbar^2\ell(\ell+1)/2mR^2 > \epsilon_0$, the conditions of Region III prevail, which would make σ_ℓ negligibly small. The neutron cross section σ_n is thus given by

$$\sigma_n = \xi\sqrt{\epsilon/\epsilon_0}\ \pi \lambda\!\!\!^{-2} \sum_{\ell=0}^{k_0 R} (2\ell+1)\, T_{\ell a}(\epsilon) \qquad \epsilon \ll \epsilon_0 \tag{63a}$$

For very small energies only the term $\ell = 0$ contributes,

$$\sigma_n = \xi\pi\lambda\!\!\!^{-2}\sqrt{\epsilon/\epsilon_0}$$

which represents the well-known 1/v-law and is identical with (34).

For high energies $\epsilon \gg \epsilon_0$, we must distinguish three groups of σ_ℓ: $(\hbar^2/2m)\left[\ell(\ell+1)/R^2\right] < \epsilon = \epsilon_0$ characterizes a region, where the energy ϵ is very much higher than the barrier and σ_ℓ is calculated by (59a). $(\hbar^2/2m)\left[\ell(\ell+1)/R^2\right] > \epsilon + \epsilon_0$ ξ falls into the Region III; σ_ℓ is negligibly small there. The intermediate region, $(\hbar^2/2m)\left[\ell(\ell+1)/R^2\right] \sim \epsilon$, includes only a relatively small number of ℓ's, so that the total result will be:

$$\sigma_n \approx \xi\sum_0^{kR}(2\ell+1)\pi\lambda\!\!\!^{-2} = \xi\pi R^2 \qquad \epsilon \gg \epsilon_0 \tag{63b}$$

For high energies the cross section approaches πR^2, since the coefficient ξ should not be far from unity.

Fig. 3 shows the theoretical prediction for the neutron cross section

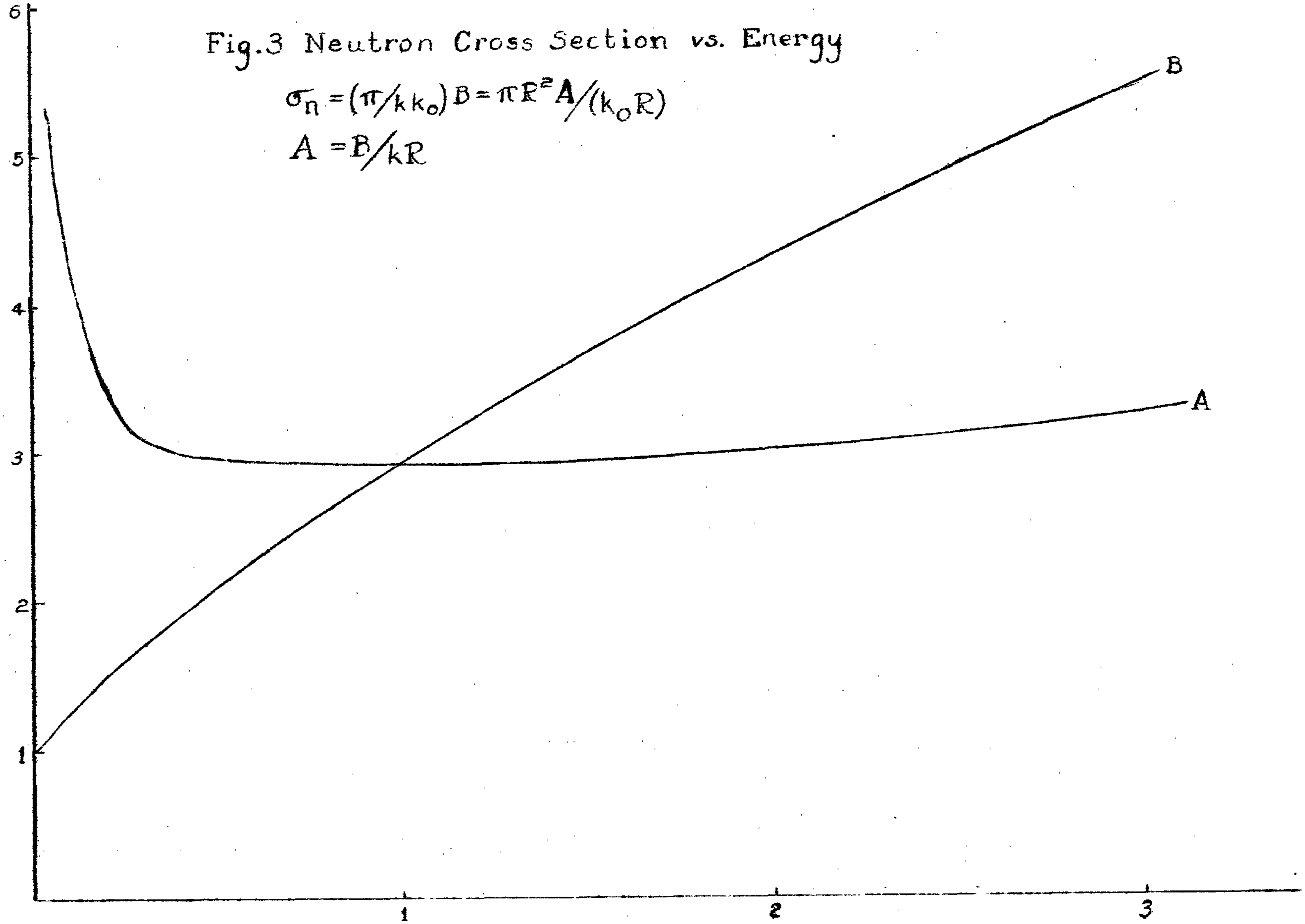


Fig.3 Neutron Cross Section vs. Energy

$\sigma_n = (\pi/kk_o)B = \pi R^2 A/(k_o R)$

$A = B/kR$

as functions of the energy ϵ. The energy is given in units of $(kR)^2$. The curve B is given by $B=\sum_{\ell=0}^{k_oR}(2\ell+1)T_{\ell a}$. This sum converges rapidly for the values of ϵ for which it is plotted so that the upper limit of the sum does not enter. B is proportional to $\sigma_n V$, and A is proportional to the cross section itself. It is seen, that, with the plausible value for $k_oR \sim 3$, the cross section σ_n goes smoothly over into its asymptotic value πR^2.

For a heavy nucleus of $A \sim 200$, the radius R is near to 9×10^{-13} cm, so that $(kR)^2$ is a measure of the energy in units of 250 Kev. Hence we must expect σ_n to follow a 1/v law for low energies up to about 130 Kev and to assume a value near $\pi R^2 \cong 3 \times 10^{-24}$ above that energy.

The cross section σ for charged particles can be computed as follows: it is given by (60) as long as their energy is appreciably lower than the barrier. The WKB method can be applied to calculate the penetration factor, if the barrier extends over a region which is large enough, so that the potential energy does not change appreciably over one wavelength of the particle. This is the case for higher nuclear charges $(Z > 20)$. If the energy is well above the barrier, the expressions (59a) give the cross sections directly, since there is then no penetration factor to calculate. For energies near the barrier, however, the expressions (59b) should be used. The WKB method in this region is no longer reliable, since the remaining barrier is necessarily very low and extends over a narrow region. In the following computations, this region has been left out completely. The validity of the WKB solution, applied at low energies, has been extended up to energies equal to the barrier, where it was joined with the solution valid at energies much higher than the barrier. These two solutions join smoothly at that energy since the WKB expression for the penetrability P of the barrier becomes unity at the barrier. The justification of this procedure lies essentially in the fact that, the barrier heights for different angular moments of the incident particle differ by amounts of the order of 1 Mev, so that, for a given energy of the incident particle, only one or two angular moments lead to a case of a Region

II. Most of the partial cross sections σ_ℓ are thus computed correctly *).

The cross section for the penetration through the barrier is a very sensitive function of the nuclear radius. The following tables give the values for two sets of nuclear radii, one given by $R = 1.5 \times A^{1/3}$, the other by $R = 1.3 \times A^{1/3}$. It seems that the second set with its steeper excitation functions, fits better to the experimental material available.

The calculation of η_b of expression (2) involves the computation of the $\overline{\Gamma}_b$ for all particles b emitted by the compound nucleus. They are given by the sum $\overline{\Gamma}_b = \sum_{\beta\ell} \overline{\Gamma}_{b\beta\ell}$ each term of which can be computed from (59a), (59b), (60) with reasonable accuracy, apart from the factor ζ which does not differ much from unity. If the final nucleus can be left in many excited states β, the sum can be expressed as an integral and we obtain from (16a):

$$\overline{\Gamma}_b = \frac{1}{2\pi^2 \omega_c(E)} \int_0^{\epsilon_{b\,max}} \frac{m\epsilon}{\hbar^2} \sigma_b(\epsilon)\, \omega_R(\epsilon_{b\,max} - \epsilon)\, d\epsilon \qquad (64)$$

$\epsilon_{b\ max}$ is the maximum energy the particle b can obtain (that is, by leaving the residual nucleus in its ground state). $\omega(E_\beta)$ is the level density of the residual nucleus, at the excitation energy E_β, $\sigma_b(\epsilon)$ is the cross section (summed over all ℓ) to form a compound nucleus of excitation E by bombarding the residual nucleus with a particle b of an energy ϵ; (the residual nucleus must have been in an excited state of an energy $(\epsilon_{b\ max} - \epsilon)$. The common factor $1/2\pi^2\omega_c$ drops out of η_b and only the pure number

$$f_b(\epsilon_{b\,max}) = \int_0^{\epsilon_{b\,max}} \frac{m\epsilon}{\hbar^2} \sigma_b(\epsilon)\, \omega_R(\epsilon_{b\,max} - \epsilon)\, d\epsilon \qquad (65)$$

need to be computed, which is a function of $_{b\ max}$ only. Then we can write

$$\eta_b = \left(f_b \Big/ \sum_c f_c \right) \qquad (66)$$

*) In a few cases the above procedure was checked successfully by an exact calculation.
This procedure differs from the calculation used in Ph.R. 57 472 (1940). It gives steeper excitation functions. It is equivalent to the expressions proposed in Bethe and Konopinski Phys. Rev. 54, 130 (1938).

x	40 Ca 20 I p	I α	II p	II α	64 Zn 30 I p	I α	II p	II α	90 Zr 40 I p	I α	II p	II α
.25	---	0	---	0	--	0	---	0	--	0	---	0
.4	2.55	0	2.41	0	.846	0	.730	0	.321	0	.308	0
.5	6.37	.0391	7.00	.0239	3.42	.00468	3.04	.00145	1.70	.00039	1.61	0
.6	12.3	.360	13.4	.298	8.15	.0996	8.23	.0793	4.94	.0333	5.77	.0223
.7	18.3	1.84	22.3	1.72	15.1	.871	16.1	.873	11.5	.451	13.8	.410
.8	26.5	5.48	31.7	6.13	23.2	3.90	26.7	4.19	20.1	2.79	24.9	2.87
.9	35.1	13.1	41.6	15.7	33.7	11.2	38.8	13.6	31.5	10.7	40.2	11.7
1.0	40.3	21.0	50.7	27.6	46.3	20.6	49.3	26.9	41.5	21.3	48.1	27.0
1.2	50.0	39.5	63.3	53.6	54.8	44.7	69.9	58.3	59.4	49.9	79.4	67.5
1.4	56.2	58.4	71.6	71.7	63.8	61.9	79.1	79.0	71.9	70.2	96.9	94.4
1.6	60.4	59.5	77.9	83.4	70.2	71.4	86.9	94.6	80.1	82.1	110	114
1.8	64.0	65.6	83.3	93.0	73.7	80.3	103.	104	86.3	92.3	117	127
2.0	66.2	72.4	87.5	102.	78.3	86.9	110	114	96.1	104	125	140
B	6.47	10.06	5.60	8.72	8.28	13.34	7.17	11.56	9.92	16.62	8.54	14.06

x	120 Sn 50 I p	I α	II p	II α	142 Nd 60 I p	I α	II p	II α	173 X 70 I p	I α	II p	II α
.25	.00393	0	---	0	----	0	----	0	----	0	----	0
.4	.149	0	.119	0	.0677	0	.0499	0	.0247	0	.0209	0
.5	1.01	0	.843	0	.625	0	.531	0	.342	0	.292	0
.6	3.85	.0111	3.96	.00730	2.92	.00431	2.88	.00284	1.92	.00167	1.90	.00105
.7	7.61	.252	11.1	.218	8.39	.135	8.79	.0991	6.64	.0806	7.49	.0680
.8	19.9	2.00	22.7	1.99	17.6	1.42	17.3	1.54	15.4	1.06	16.1	1.08
.9	32.9	8.91	39.9	10.7	32.0	7.85	38.1	9.50	29.9	7.15	36.3	8.19
1.0	45.4	20.8	56.0		45.2	21.4	88.8	26.5	44.6	20.7	56.4	26.7
1.2	68.7	56.2	87.1	74.6	67.0	59.9	91.6	78.8	74.4	65.3	95.2	85.1
1.4	84.1	79.3	108	106	89.1	87.0	115	112	94.6	94.5	120	124
1.6	95.4	93.5	122	128	102	105	137	137	108	115	139	149
1.8	103	106	133	145	111	119	141	157	117	130	152	172
2.0	110	118	141	159	119	131	150	172	125	143	163	189
B	11.20	18.73	9.70	16.24	12.67	21.45	10.98	18.59	13.96	23.68	12.10	20.52

where the sum in the denominator is extended over all particles c which can be emitted in this reaction. It is evident that the integrand in (64)

TABLE II

The cross sections σ_p and σ_α are given for 8 typical elements as function of the energy ϵ :

$$\sigma_{p,\alpha} = n(x) \cdot 10^{-26}\ cm^2$$

n is the number given in the tables, x is a measure of the energy in units of the barrier height B given in each column: $\epsilon = x \cdot B$. The columns headed by I correspond to a nuclear radius $R = r_o A^{1/3}$ with $r_o = 1.3 \times 10^{-13}$ cm, the columns headed by II, correspond to $r_o = 1.5 \times 10^{-13}$ cm.

The cross section $\sigma_d(\epsilon_d)$ for the penetration of the deuteron is equal to the one of the proton for an element with a charge smaller by a factor $(\frac{1}{2})^{4/3}$ and with an energy smaller by $(\frac{1}{2})^{\frac{1}{2}}$.

	$_{80}Hg^{201}$				$_{90}Th^{232}$			
	I		II		I		II	
x	p	α	p	α	p	α	p	α
.25	----	0	---	0	----	0	---	0
.4	.0120	0	.00959	0	.00265	0	.00420	0
.5	.217	0	.181	0	.0562	0	.111	0
.6	1.46	.00075	1.40	0	.471	0	1.03	0
.7	5.80	.0522	6.01	.0419	2.23	.0315	5.02	.0238
.8	14.3	.881	16.2	.840	6.18	.651	15.1	.666
.9	29.9	7.01	35.3	7.81	14.9	6.08	36.1	7.02
1.0	46.5	22.7	56.7	27.2	26.1	21.2	57.7	26.5
1.2	75.8	69.8	99.1	91.1	54.4	72.6	107	95.6
1.4	100	100	129	132	76.4	107	137	141
1.6	113	123	149	162	92.1	132	160	173
1.8	126	141	163	184	----	150	176	198
2.0	138	155	---	205	----	167	---	219
B	15.18	25.93	13.16	22.47	16.28	28.00	14.15	24.27

$$I(\epsilon)d\epsilon = (2m/\hbar^2)\,\epsilon\,\sigma_b(\epsilon)\,\omega_R(\epsilon_{b\,max} - \epsilon)\,d\epsilon \qquad (67)$$

is the probability of emission of b with an energy between ϵ and $\epsilon + d\epsilon$. It describes the energy distribution for the outgoing particle as a function of its energy ϵ . (This is identical with expression (4)).

It is interesting to discuss the connection between the distribution (67) and the Maxwell distribution of particles evaporating from a hot sphere, as indicated in the introduction.

The distribution $I(\epsilon)$ can be written

$$I(\epsilon) = (2m/\hbar^2)\epsilon\,\sigma_b\,\omega_R(\epsilon_{b\,max} - \epsilon) = (2m/\hbar^2)\epsilon\,\sigma_b \exp\left\{S(\epsilon_{b\,max} - \epsilon)/k\right\}$$

where $(1/k)S(E_\beta)$ is the logarithm of the level density of the residual nucleus. According to statistical mechanics, $S(E)$ is the entropy of the residual nucleus with the energy E. By writing

$$S(\epsilon_{b\,max} - \epsilon) = S(\epsilon_{b\,max}) - \epsilon(\partial S/\partial E)_{\epsilon_{b\,max}}$$

and by using the well-known statistical expression

$$\partial S/\partial E = 1/T \tag{68}$$

we obtain

$$I(\epsilon) = \text{const.}\,\epsilon_b\,\sigma_b\,e^{-\epsilon_b/kT} \tag{69}$$

Here T is the temperature at which, according to (67), the residual nucleus has the average energy $\epsilon_{b\,max}$. If σ_b is a slowly varying function of the energy, as it is in the case of the emission of neutrons, the emitted particles have a Maxwellian energy distribution, just as particles would have, that are evaporated by a material with a temperature T. This temperature is, however, not the "temperature of the compound nucleus before emission", but the "temperature of a nucleus after emission". Here we understand by "temperature of a nucleus in a state of energy E" the temperature at which it has an average excitation energy E. This comes from the fact that the "evaporation" of one particle constitutes a relatively large loss of energy which reduces the temperature considerably. The Maxwell distribution of the emitted particles is determined by the temperature <u>after</u> the emission.

If σ_b is a strongly varying function of the energy, as it is in case of charged particles emitted, the Maxwell distribution is deformed by the factor σ_b in (68). Since σ_b is usually a strongly increasing function for charged particles, the maximum of the distribution will be shifted to higher energies.

The limitations of the validity of the Maxwell distribution in nuclear emissions are obvious from the method of derivation: it is assumed (a) that the level density of the residual nucleus in the region of possible excitation is high enough to permit statistical treatment, (b) that $\epsilon \ll \epsilon_{b\ max}$, which means that the energy of the emitted neutron is small compared to its maximum available energy. Since the level density of nuclei is unknown to a great extent, it is hard to predict the limits of validity in terms of Mevs. In recent experiments*) it seems that the level density of the lower lying levels is rather small, viz., about one or two levels per Mev in the first or second Mev of excitation energy. The maximum energy $\epsilon_{b\ max}$ probably must be at least 5 - 7 Mev in order to expect a shape similar to a Maxwellian distribution for the emitted neutrons by elements in the middle or at the heavy end of the periodic system.

The values of the temperature T, which are expected in nuclear reactions, can be estimated in the following way: the average energy E of a system is a monotonically increasing function of the temperature T. If the temperature is very high, so that all degrees of freedom are excited, E is proportional to T. This is certainly not the case in a nucleus, when it is excited by a normal nuclear reaction. If one considers the nucleus very roughly as a gas of A particles concentrated in a volume ($\frac{4\pi}{3} R^3$), one finds that this "gas" is degenerated if the excitation energy is of the order or less than $\sim A\hbar^2/2MR^2 \cong 9A^{1/3}$ Mev. Heavy nuclei should therefore be considered as highly degenerated "gases". The strong interaction between the particle does not affect this conclusion which also holds for an electron gas in metals in spite of their

*) Wilkins, T. R. and Kuerti, G., Phys. Rev. 55, 1134 (1939).

Wilkins, T. R., Phys. Rev. 60, 365 (1941).

Dicke, R. H. and Marshall, J., Phys. Rev. 63, 86 (1943).

Dunlap, H. F. and Little, R. N., Phys. Rev. 60, 693 (1941).

interaction.

The function E(T) has a vanishing derivative: $(\partial E/\partial T)_{T=0} = 0$ for $T = 0$. (This means that the specific heat is zero for $T=0$ according to the third law of thermodynamics.) The expansion of E(T) near $T=0$ starts therefore with a quadratic term. If a system is highly degenerate, we may write

$$E = bT^2 \tag{70}$$

and neglect the higher powers of T, which are small in this case. From this we get

$$S = \int \frac{dE}{T(E)} \sqrt{aE} + \text{const.}, \quad a = 4/b$$

and finally for the level density:

$$\omega(E) = C\, e^{\sqrt{aE}} \tag{71}$$

One can try to adjust the constants a and C, so that this equation reproduces our rather scant knowledge on level densities. A fair choice for $A > 60$ is e.g.,

$$C_{odd} = \left[12/(A-40)\right] \ (\text{Mev})^{-1}, \quad C_{even} = \tfrac{1}{2} C_{odd}, \quad a = 3.35(A-40)^{\frac{1}{2}} \ (\text{Mev})^{-1}$$

if the excitation energies E are measured in Mev. C_{odd} and C_{even} refers to nuclei with odd or even A respectively.

The expression (71) is adjusted to our knowledge of level densities for very low excitation energies and for excitations in the region of capture of slow neutrons. In the latter case only levels, whose spin differs by $\pm\frac{1}{2}\hbar$ from the initial nucleus are observed. The level density in the expression (71), however, should include all levels which can emit or absorb the particles b with the energy ϵ_b. Our knowledge of level densities and of the distribution of angular momenta among the levels is insufficient to make any distinction of that kind at the present time.

From (70) an estimate can be given of the nuclear temperature which appears in expression (69) and which determines the Maxwell distribution of the outgoing neutrons. If the mazimum possible energy of the outgoing neutrons is E_0, the temperatures are given in the following table in Mev's according to

$1/T = \left(\frac{\partial S}{\partial E}\right)_{E=E_0}$

E_0 \ A	60	150	230
5	1.15	.76	.66
10	1.63	1.07	.93
15	2.00	1.31	1.14

These temperatures can be obtained by bombarding the nucleus with neutrons of an energy E_0, or by any process leading to the same excitation energy of the compound nucleus. Expression (69) shows that the maximum intensity is emitted with an energy $2kT$, if σ is a slowly varying function of ϵ, as it is expected to be in case of neutron emission.

With the expression (71) for the level density it is possible to calculate the functions $f(\epsilon_{bmax})$ defined in (65) which serve to compute the factor η_b according to (65). Examples are shown in Fig. 4.

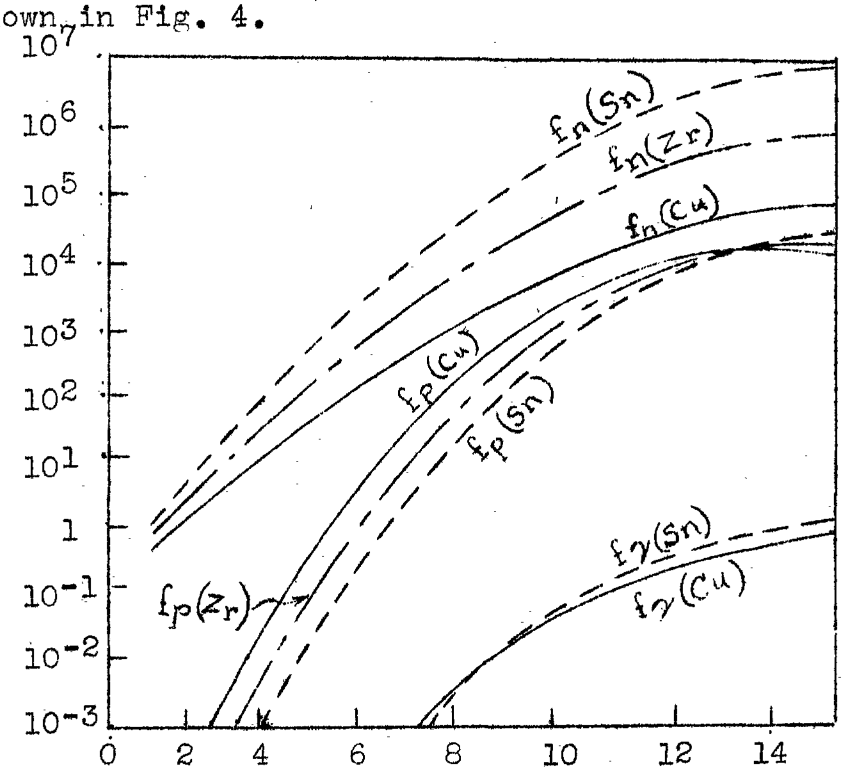


Fig.4. The functions f_n, f_p and f_γ are given as functions of energy for the compound nuclei Cu, Zr and Sn. These values must be multiplied by 2 or 0.5 if the residual nucleus is odd-odd or even-even, respectively. Expression (71) for the level density is used in the computation.

B) Discussion of Special Reaction Types

The reaction types, discussed in this section, are ordered according to the nature of the incoming particle. Resonance reactions are excluded, since they were discussed in Section V.

1. Neutron Reactions

The cross section $\sigma_n(\epsilon)$ for the formation of a compound nucleus for energies ϵ above the resonance region is given by (63a) and (63b), and is represented in Fig. 2.

(n,n) Reactions. This is not a real nuclear reaction since the bombarded nucleus remains unchanged. It may however be left in an excited state after reemission of the neutron, in which case we speak of inelastic scattering of neutrons by the nucleus. Once the compount nucleus is formed, every other particle except the neutron has to penetrate a potential barrier to get out. The neutron width, therefore, is much larger than any other particle width, and also larger than the radiation or fission width if ϵ is sufficiently high. Under these conditions $\eta_n \sim 1$ and the cross section for the (n,n) reaction is given by σ_n itself.

There is a difficulty in interpreting the observed (n,n) cross section by means of the expression (2), due to the presence of elastic scattering. This is the part of the n,n reaction in which the outcoming neutron has the same energy as the incoming one. As mentioned in Section XXXII, page 8, part of the elastic scattering is due to effects in which the neutron is deflected by the nuclear potentials, and thus does not penetrate and form a compound nucleus. This part of the scattering is evidently not included in the expression (2). The observed cross section of an (n,n) reaction therefore contains a part which has nothing to do with the formation of a compound nucleus. The waves of the scattered neutrons due to external effects (in which no compound nucleus is formed) combine coherently with the neutrons, emitted after the formation of the compound nucleus if they are re-emitted by leaving the nucleus in its original state. These two effects together give the elastic scattering.

The observed cross section of an (n,n) reaction therefore contains a part which has nothing to do with the formation of a compound nucleus. The observed $\sigma(n,n)$ can be much larger than σ_n, especially at low energies, when the elastic scattering is a relatively large part of the total scattering. It is, of course, possible to separate the elastic from the inelastic scattering by distinguishing between the scattered neutrons of different energy. The cross section σ_{in} for inelastic scattering, which is due to the neutrons which have left the nucleus in an excited state, is certainly lower than σ_n, because it does not contain the contribution from the elastic scattering, which corresponds to a real penetration into the nucleus and a subsequent re-emission by the compound nucleus with the same energy as it came in.

The part of the elastic scattering coming from the real formation of the compound state could be determined by creating the same compound nucleus in the same state of excitation by means of any other reaction. This compound nucleus emits neutrons exactly in the same way as it would do if created in a (n,n) reaction, according to Bohr's assumption B, but it does not contain the scattering without formation of a compound state. The relative amount of neutrons, which leave the residual nucleus in the ground state, can then be determined. This amount corresponds to the part of the elastic scattering which comes from the true nuclear reaction.

The energy of the emitted neutrons is given by the level spectrum of the bombarded nucleus. If the energy of the incoming neutron is less than the lowest excitation energy of the nucleus, only elastic scattering occurs. If the primary energy is higher, inelastic scattering can occur; the energy loss is always equal to an excitation level of the nucleus. If the primary energy is so high that a great number of levels can be excited, the energy distribution is given by expression (67) and becomes similar to a Maxwell distribution, as described on page

There is only very little experimental material available on the spectrum of the scattered neutrons by heavier elements. Dunlap and Little*) measured the spectrum of neutrons scattered by Pb with an initial energy of 2.5 Mev and found an inelastic scattering which indicates several excitation levels of Pb.

The <u>(n,2n)-reaction</u> occurs if the energy of the incident neutron is sufficiently high, the residual nucleus after the re-emission of the incident neutron, may still be excited highly enough to emit a second neutron. This is possible only if $\epsilon_n - \epsilon_b$ is larger than the binding energy B_n of a neutron to the bombarded nucleus where ϵ_n is the energy of the incident neutron and ϵ_b is the energy of the first emitted neutron. The probability of (n,2n) reaction is therefore given by:

$$\sigma(n,2n) = \sigma_n \frac{\int_0^{\epsilon_n - B_n} I(\epsilon)\,d\epsilon}{\int_0^{\epsilon_n} I(\epsilon)\,d\epsilon}$$

where $I(\epsilon)d\epsilon$ is the probability of emission of a neutron with the energy ϵ, given by (67). For energies ϵ_n which make a (n,2n) reaction possible, the energy distribution can be well approximated by a Maxwell distribution (69) with a temperature T, with which one obtains (with $\sigma_n = \xi\pi R^2$):

$$\sigma(n,2n) = \xi\pi R^2\left[1-\left(1+\Delta\epsilon/T\right)e^{-\Delta\epsilon/T}\right] \qquad (71a)$$

$$\Delta\epsilon = \epsilon_n - B_n$$

If $\Delta\epsilon$ -- the energy surplus over the threshold energy of the reaction -- is large compared to the temperature T, the (n,2n) reaction should be the dominant reaction and its cross section is then nearly equal to σ_n. The (n,n) reaction cross section should become correspondingly smaller. This should occur for $\Delta\epsilon \gtrsim 2$ or 3 Mev for elements in the middle of the periodic systems. For very high primary energies -- $\epsilon > 2B_n$ -- (n,3n) reactions will occur.

*) Dunlap and Little, Phys. Rev.

The <u>(n,p) reactions</u> naturally are less probable than the (n,n) reactions, because of the potential barrier which prevents the proton from leaving the nucleus as easily as a neutron. The cross section is given by:

$$\sigma(n,p) = \sigma_n \frac{f_p}{f_n + f_p}$$

where the f's are given by (65). This cross section assumes measurable values only if the energy of the incident neutron is several Mev above the reaction threshold, because the outgoing proton needs that much energy to penetrate the barrier. The rather steep dependence of $\sigma(n,p)$ on the energy gives rise to an apparent threshold which is much higher than the actual threshold.

If the nucleus which is created by the (n,p) reaction is a negative electron emitter, the final product of the β-decay is identical with the initially bombarded nucleus. In this case the reaction energy $Q = \epsilon_p - \epsilon_n$ of the (n,p) reaction can be calculated by the energy law. The energy balance in the nuclear reaction

$$n + X = Y + p$$

is given by

$$\epsilon_n + E_X = E_Y + \epsilon_p$$

where E_X and E_Y are the binding energies of the nuclei X and Y in their ground states. The β-decay has the following energy balance:

$$E_Y = E_X - m_n + m_p + \epsilon^- + h\nu$$

where m_n and m_p are the mass energies of the proton and the neutron and ϵ^- includes the mass energy mc^2 and the kinetic energy ϵ^-_{kin} of the electron emitted; $h\nu$ is the energy of a γ-ray which may accompany the β-decay. We thus get for the Q-value:

$$Q = \epsilon_p - \epsilon_n = m_n - m_p - \epsilon^- - h\nu = 0.7 \text{ Mev} - h\nu - \epsilon^-_{kin} \qquad (72)$$

<u>(n,α) reactions</u> have an extremely small cross section in heavy nuclei because of the potential barrier, which makes it very improbable that the α-particle leaves the nucleus. If the neutron energies are extremely high,

however, (n,α) reactions may occur in heavy nuclei. There will be more chance to observe these reactions at very heavy elements, where the binding energy of the α-particle is smaller.

The (n,γ) reaction is usually called radiative neutron capture and was discussed extensively in the resonance region in Section XXXV. In the higher energy region the average value over many resonances is observed.

$$\sigma(n,\gamma) = \sigma_n(\overline{\Gamma}_r/\Delta)$$

The processes which compete in heavy elements with the radiation, are the re-emission of the neutron and, in some elements, the nuclear fission. We exclude the latter from these considerations. The cross section for the radiative capture of a neutron with an angular momentum $\ell\hbar$ is given by

$$\sigma_\ell(n,\gamma) = \sigma_\ell^{(n)} \frac{\overline{\Gamma}_r}{\overline{\Gamma}_r + \overline{\Gamma}_n^\ell} \tag{73}$$

$\overline{\Gamma}_r$ is the average radiation width and $\overline{\Gamma}_n^\ell$ is the neutron width for the compound states, which are excited by a neutron with an angular momentum $\ell\hbar$. If the primary neutron energy is not too high (less than 1 Mev), the probability of inelastic scattering is relatively low, so that the re-emission of the neutron is essentially the process inverse to the capture*). Γ_n^ℓ stands then in the relation (16) to $\sigma_\ell^{(n)}$, and we get for the total capture cross section:

$$\sigma(n,\gamma) = \frac{2\pi^2 \lambda\!\!\!^{-}{}_n^2}{D} \sum_\ell \frac{\Gamma_r}{1 + \Gamma_r/\Gamma_n^{(\ell)}} \tag{73a}$$

This can be calculated with the help of (59b). For small energies ($\lambda\!\!\!^{-} \gg R$) only $\ell = 0$ contributes and we obtain from (73) and (59b) by assuming $\Gamma_r \gg \Gamma_n^{(o)}$

$$\sigma(n,\gamma) = \zeta\pi\,(k_n/k_o)\,\lambda\!\!\!^{-}{}_n^2 = \zeta\pi\,\lambda\!\!\!^{-}{}_n\,\lambda\!\!\!^{-}{}_o \tag{74}$$

where $\lambda\!\!\!^{-}{}_o = k_o^{-1}$ and $\lambda\!\!\!^{-}{}_n = k_n^{-1}$ is the wave length of the incoming neutron. With a value of $\lambda\!\!\!^{-}{}_o$ corresponding to 2 Mev, one obtains

*) If the angular momentum of the bombarded nucleus is different from zero, the neutron can be re-emitted with an angular momentum ℓ' different from the angular momentum ℓ of the captured one. The parity rule allows only $|\ell-\ell'| =$ 2, 4 etc., and a closer investigation shows that the probability of $\ell-\ell' \neq 0$ is rather small.

$$\sigma(n,\gamma) = \zeta(44Q/\sqrt{\epsilon}) \text{ for } \epsilon < 10^4$$

if ϵ is measured in ev. ζ is a magnitude which should be not far from unity.*)

An example of the energy dependence of $\sigma(n,\gamma)$ is given in Fig. 5, which shows expression (73a) with the following choice of constants: $R = 9 \times 10^{-13}$, $k_0 R = 3$, $D/2\pi = 45\,\Gamma_r$, $\zeta = 1$.

$\sigma(n,\gamma)$ becomes very small if strong inelastic scattering sets in, since the value of $\overline{\Gamma}_n$ is then considerably increases. This may happen at energies as low as 0.5 Mev in some elements. In others (73a) may be valid up to 1.5 Mev.

2. Proton Reactions

The theoretical cross section $\sigma_p(\epsilon_p)$ for the formation of a compound nucleus by a proton is given in Table II for some characteristic elements. The actual values are probably somewhat smaller because of the factor ζ.

The most probable proton reaction is the (p,n)-reaction. Only in cases where the proton energy is not high enough to emit a neutron, other reactions have a chance of a considerable cross section. The reaction energy Q of a (p,n) reaction can also be determined from the energy of the β-rays of the nucleus created, if the latter is a positron emitter. One obtains instead of (72):

$$Q = \epsilon_n - \epsilon_p = m_p - m_n - h\nu - \epsilon^+ = - h\nu - \epsilon^+_{kin} - 1.7 \text{ Mev}$$

Thus the Q-values of (p,n) reactions are usually negative and of the order of several Mev's.

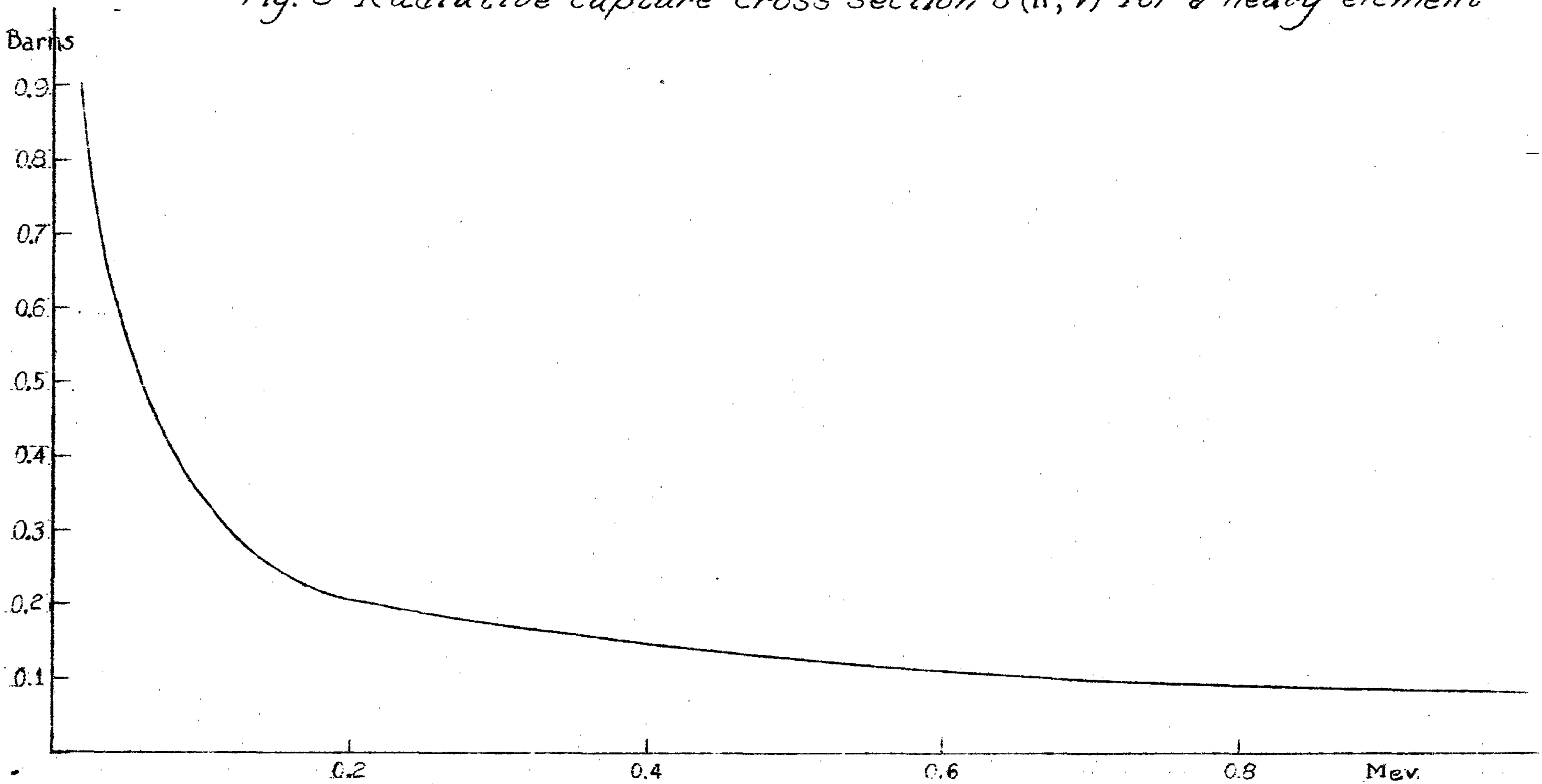
The cross section of a (p,n) reaction is given by:

$$\sigma(p,n) = \sigma_p \eta_m \tag{75}$$

and η_n is very near unity above the threshold apart from exceptional cases in which the energy of the outgoing neutron is very much smaller than the energy of proton. This may happen just above the threshold if the threshold energy is very

*) Note that this is an average over a region containing many levels and thus does not apply to thermal energies.

Fig. 5 Radiative capture cross section $\sigma(n,\gamma)$ for a heavy element



high. Then the advantage of the neutron emission, which comes from the absence of a barrier is made up by the high energy of the competing proton re-emission. In this case η_n can be smaller than 1 and is determined by $\eta_n = f_n/(f_n+f_p)$. The f's are functions of the maximum energy ϵ_{max} of the respective particles as described previously. If $\epsilon_{n\ max} \ll \epsilon_{p\ max}$, it may happen that $f_p > f_n$, particularly if the nuclear charge Z is not too high. (See Fig. 4.) A case of this sort is found in the Ni^{62} (p,n) Cu^{62} reaction which has a threshold of 4.7 Mev*).

In general, however, the cross section of a (p,n)-reaction should be very closely equal to σ_p in Table II for different nuclei. Comparison with experiments*) have shown that a nuclear radius $R = r_o A^{1/3}$ with $r_o \sim 1.3 \times 10^{-8}$ cm fits better for elements in the middle of the periodic system between Ca and Ag.

The energy distribution of the emitted neutrons, is similar to the distribution in an (n,n)-reaction. The highest possible energy occurs when the final nucleus is left in its ground state, and the emitted spectrum corresponds to the excitation spectrum of the final nucleus in the way that lower energies in the spectrum differ from the highest by an amount equal to an excitation energy of the nucleus. For high initial proton energies, this spectrum becomes similar to a Maxwell distribution.

If the maximum energy $\epsilon_{n\ max}$ of the neutrons of a (p,n) reaction is larger than the binding energy of a neutron to the final nucleus, the latter can be left in an excited state high enough to emit a second neutron. We then obtain a <u>(p,2n)-reaction</u>. The probability of this event is given by a similar expression to (71a); σ_n has to be replaced by σ_p. The energy distribution $I(\epsilon)$ of the emitted neutrons can be represented here also by a Maxwell distribution and we get:

$$\sigma(p,2n) = \sigma_p\left[1-(1+\Delta\epsilon/T)\,e^{-\Delta\epsilon/T}\right] \qquad (76)$$

*) Weisskopf and Ewing, Phys. Rev. <u>57</u>, 472, (1940).

$\Delta\epsilon$ is the excess energy of the proton over the (p,2n) threshold.

The (p,γ) reaction is the most important reaction in case the proton energy is below the (p,n) threshold. The only competing process is then the re-emission of the proton [(p,p) reaction], which is not probable since the proton has to penetrate the barrier again. The cross section is given by:

$$\sigma(p,\gamma) = \sigma_p \frac{f_\gamma}{f_p + f_\gamma + f_n} \tag{77}$$

and is vanishingly small above the (p,n) threshold because f_n rises rapidly and becomes larger than all other f's. No resonance effects should be expected in the (p,γ) reactions since in heavy elements, in the resonance region, the energy of theof the proton is much too small to penetrate the barrier to an observable extent. The (p,p) reaction can be observed if the proton energy is not sufficient to emit a neutron with appreciable probability. Since the Rutherford scattering by the Coulomb field is always present and usually much stronger than any nuclear reaction, a (p,p) reaction can only be observed either by registering scattered protons with less energy than the incoming ones*) or by finding the bombarded nucleus in an excited state**) which must be a metastable (isomeric) state in order to be observed.

The cross section of a (p,p) reaction is given by:

$$\sigma(p,p) = \sigma_p \frac{f_p}{f_p + f_\gamma + f_n}$$

It rises steeper than σ_p with increasing energy ϵ_p of the incoming proton, as long as $f_p < f_\gamma$, since f_p also is a strongly rising function of ϵ_p. $\sigma(p,p)$ is then essentially proportional to the square of the penetration of the barrier.

3. α-Particle Reactions

These reactions are very similar to the proton reactions. The (α,n)

*) R. Wilkins, Phys. Rev. 60, 365 (1941).
R. Dicke and J. Marshall, Phys. Rev. 63, 86 (1943).

**) S. Barnes, Phys. Rev. 56, 414 (1939).

reaction is the most probable one and its cross section is given by an expression like (75) with σ_{α} instead of σ_{p}. η_n is almost unity except below or near the threshold energy. The $(\alpha,2n)$ reaction is given by the same expression as (76) after replacing σ_p by σ_{α}, and so is the (α,γ) reaction cross section given by (77). The latter one is unobservably small since, below the (α,n) threshold, the α-particle hardly can penetrate the barrier.

The (α,p) reaction has a cross section of the form

$$\sigma(\alpha,p) = \sigma_{\alpha} \frac{f_p}{f_p + f_n}$$

and is always smaller than the (α,n) reaction but can have observable cross sections especially with low Z and $\epsilon_{p\ max} > \epsilon_{n\ max}$.

It is hard to estimate the threshold energies for the different α-reactions, because of the fact that the binding energy of the α-particle to the compound nucleus is not well known, but is needed to determine the excitation energy of the compound nucleus. The binding energy B_{α} can be estimated as follows: four nuclear particles are added, which have had in the α-particle a binding energy of 28 Mev. Thus the binding energy of the α-particle should be $B_{\alpha} = 4B - 28$ Mev, where B is the average binding energy of the two protons and two neutrons added. B is not well enough known to make any guess about B_{α}. Recent results on $(\alpha,2n)$ reactions*) have shown that, with 15 Mev α-particles, $(\alpha,2n)$ reactions were not observable. This shows that, at least in the investigated elements: $B_{n_1} + B_{n_2} - B \geq 15$ Mev, where $B_{n_{1,2}}$ are the binding energies of the first and second neutron of the $(\alpha,2n)$ reaction.

4. Deuteron Reactions

The d-reactions follow in general the same laws as the proton reactions. They distinguish themselves from all other reactions because of the very high excitation of the compound nucleus. The excitation energy is equal to the kinetic energy of the deuteron plus the binding energies of two particles reduced by the

*) R. N. Smith, Dissertation, Purdue University, unpublished

relatively small internal binding energy of the deuteron of 2.15 Mev. It amounts to about 14-16 Mev plus the kinetic energy. This is one of the reasons why deuteron-reactions have a relatively large yield.

The (d,n) and (d,2n) reactions obey the same laws as (75) and (76). σ_d is naturally smaller than σ_p at the same energy. The maximum energy $\epsilon_{n\ max}$ of the outgoing neutron is usually much higher than the deuteron energy because of the high excitation of the compound nucleus. Thus, there is no positive threshold for the (d,n) reaction, and η_n is always nearly unity.

The threshold of the (d,2n) reaction can be calculated from the β-energy of the produced radioactive nucleus if it is a positron emitter. According to the same principles as in (72), we find

$$\begin{aligned} \text{(d,2n) threshold } &= \epsilon^+ + h\nu + m_n - m_p + \epsilon_D \\ &= \epsilon^+_{kin} + h\nu + 2.85 \text{ Mev} \end{aligned}$$

Here ϵ_D = 2.15 Mev is the internal binding energy of the deuteron.

The (d,p) reaction should be much less probable than the (d,n) reaction. Actually, however, (d,p) reactions are observed with equal yield as (d,n) reactions. This has been explained by Oppenheimer and Phillips*) and is due to the following process: the distance between the proton and the neutron within the deuteron is relatively large, namely $\sim 3.5 \times 10^{-13}$ cm and its binding force ϵ_D of 2 Mev is unusually small. This is why the deuteron becomes strongly "polarized" when it approaches a nucleus; the proton goes to the farther end and the neutron to the nearer end. If the neutron arrives at the nuclear surface, the proton is still some distance away and has still a potential barrier to penetrate. The nuclear forces, which then act upon the neutron are much larger than ϵ_D, and the deuteron is likely to break up, when the proton still is relatively far away from the surface. After breaking up, the proton is only under the influence of the repulsive Coulomb force and leaves the nucleus. The actual

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*) Phys. Rev. 48, 500 (1935).

nuclear reaction is a capture of a neutron after the deuteron has been broken up into its parts by the Coulomb field of the nuclear charge. The apparent effect, however, is a (d,p) reaction.

The cross section of the Oppenheimer-Phillips process is much larger than σ_d, since the latter is given by the probability that the charge reaches the nuclear surface, whereas, in the Oppenheimer-Phillips process, it reaches a point some distance outside of the surface.

(d,γ) processes do not occur with measurable yield since the compound nucleus always is able to emit a neutron with considerable energy.

5. Radiative Processes

Nuclear reactions can be initiated by the absorption of a γ-ray. It is evident that the energy $h\nu$ must be larger than the lowest binding energy of a particle. The compound state is, in this case, a highly excited state of the initial nucleus. The levels are then so close that no resonance can be expected. The cross section for the different processes is given by

$$\sigma(\gamma,b) = \sigma_\gamma (f_b / \sum_a f_a$$

where b is the emitted particle and the sum is extended over all competitors. The most common reaction of this type is the (γ,n) reaction. (γ,p) reactions are possible, but are much weaker. Any other particle would not be able to compete with either proton or neutron emission. Exceptions to this are found in very heavy elements where photo-fission occurs. The excited state is then unstable against splitting into two fragments.

The value of σ_γ is connected with the transition probabilities between nuclear states, and before describing its properties, it is necessary to discuss other radiative nuclear processes.

Radiative transitions between low-lying excited states have been observed in great number in the γ-ray emissions accompanying β-decay and α-decay. In analyzing term systems, it was found out, that, transitions occur between states which differ by two units of angular momenta, $\Delta j = 2$ and not only between states differing by one or no unit, as it is the case with atomic spectra. It is known that the change of two units corresponds to a quadrupole radiation which should be smaller in the ratio $(R/\lambda)^2$ compared to the dipole radiation emitted in the transitions $\Delta j = 1, 0$. This ratio is about 1000 in the observed region of $h\nu \sim 5 \times 10^5$ ev whereas the observed quadrupole transitions were about equally probable as the dipole transitions. This has been explained by the fact that, dipole radiation only occurs if the center of charge is moving relative to the center of mass, whereas quadrupole and higher pole radiation is emitted also in motions where the two centers stay coincident. In a nucleus, the charge is nearly evenly distributed over the mass, since protons and neutrons are not distinguished in their nuclear motion. Thus the center of mass and charge stay practically together and this strongly reduces the dipole radiation. According to the observations, this reduction is strong enough to reduce it to the strength of the quadrupole radiation.

We therefore expect that the absorption of γ-rays by nuclei, also follows the law of quadrupole or dipole absorption. The ratio of the transition probabilities in quadrupole and dipole radiation is proportional to ν^2, and we expect therefore mainly quadrupole transitions for the absorption of γ-rays of more than 10 Mev which lead to (γ, b) processes (b may be any particle), since both types of radiation were equally probable for less than 1 Mev.

Quadrupole radiation corresponds in the particle picture of light, to the emission or absorption of a photon with an orbital angular momentum of one unit, whereas in the dipole radiation, the photon only possesses an intrinsic "spin" of one unit and no orbital momentum. Thus quadrupole absorption corresponds to the absorption of a p-particle.

The cross section σ_γ and its energy dependence can then be estimated. We use the fact that the wavelength λ, even for energies as high as 30 Mev is still larger than nuclear dimensions. The absorption probability should then be just proportional to the intensity $\mathcal{E}^2$ of the electric field $\mathcal{E}$ at the nucleus. (The same principle, namely that the probability should be proportional to the neutron density at the nucleus, leads to the 1/v law for neutron capture.) This intensity for a photon is, firstly, proportional to the frequency, since $\mathcal{E}^2 \sim h\nu$. Secondly, because of the fact that it is a "p-photon" with an angular momentum one, its intensity at the nucleus is proportional to $(R/\lambda\!\!\!{}^-)^2$ (cf. page 7, Section XXXIV).

Thus the probability of absorption is proportional to ν^3. The cross section σ_γ is 1/v times the probability which gives here, because of v = c:

$$\sigma_\gamma = C(h\nu)^3$$

where C is a constant*).

This relation is found to be well fulfilled according to (γ,n) experiments made by Bethe and Gentner**) who bombarded a number of elements with γ-rays of 17 Mev and of 12 Mev. The constant C shows surprisingly little change over the periodic system and is $C = 0.65 \times 10^{-29}$ cm^2/(Mev)^3. This value can also be brought into connection with the observations on γ-rays between low-lying elements: by applying expression (16), the width for the emission of a γ-ray can be found:

$$\Gamma_\gamma = \frac{C(h\nu)^3}{\pi^2 \lambda\!\!\!{}^{-2}} D = \frac{C(h\nu)^5}{\pi^2 c^2 \hbar^2} D \tag{78}$$

D is the level distance in the region of the upper level. If this is applied to low-lying levels of $h\nu \sim 1$ Mev with a level distance of $D \sim 0.5$ Mev we obtain $\Gamma_\gamma \sim 10^{-3}$ ev. This is in good agreement with the known transition probabilities among low-lying nuclear levels+).

The radiation width Γ_r of levels of the compound nucleus created by

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*) Weisskopf, Phys. Rev. 59, 318, (1941).

**) Bethe and Gentner, Z. fur Phys. 112, 45, (1939).

) H. A. Bethe, Rev. Mod. Phys. 9, 229 (1937), see also experiments by Waldman, Phys. Rev.

neutron absorption is a composite magnitude and cannot be compared directly with Γ_γ. Γ_r represents the sum of all transitions from the level of the energy E to lower lying levels, whose number is very high. Formula (78) gives the radiation width for a transition from an excited state to the ground state only. If one applies (78) also to transitions to excited states, the radiation width Γ_r is given by:

$$\Gamma_r = \int_0^{E/\hbar} \Gamma_\gamma(\nu)\,\omega(E - h\nu)\,\hbar\,d\nu \tag{79}$$

where Γ_γ is the width as a function of the emitted radiation and ω is the level density and E the excitation energy of the level, whose Γ_r is calculated. If E is assumed 9 Mev and the expressions (71) are used for the level densities ($A \sim 100$) one obtains from (79): $\Gamma_r \sim 0.3$ Mev which is rather close to the observed values in Table I, but somewhat large.

The application of (78) to transitions to excited states is questionable. One is inclined to assume from the forms of (78) that the magnitude Γ/D is slowly variable where D is the level distance, if one varies either the upper level or the lower level of the transition. In this case one would get instead of (79):

$$\Gamma_r = \int_0^{E/\hbar} \Gamma_\gamma(\nu)\,\omega(0)\,h\,d\nu \tag{80}$$

where $\omega(0)$ is the level density near the ground state, which is of the order of 3.5 $(\mathrm{Mev})^{-1}$. One then gets for $E \sim 9$ Mev: $\Gamma_r = 2.0 \times 10^{-4}\,D'$ where D' is the level distance at the upper level, which is of the order of 40 ev. This gives definitely too small values. The truth lies probably somewhere between (79) and (80).

The dependence of Γ_r on the excitation energy E is given by (80) as proportional to E^6 and by (79) as very slowly increasing function of E. Hence it is very difficult to predict the dependence of Γ_r on the excitation energy.

The expected spectrum of the emitted radiation from a state with the excitation energy E depends also on the assumptions made. The intensity $I(\nu)d\nu$

as a function of frequency is given by the integrand of (79) or (80). If (80) is valid, the distribution is proportional to ν^5 and has a maximum at high energies; the transitions to the lowest states are the most probable ones. If (79) is valid, the intensity has a maximum somewhere at intermediate energies; it is proportional to ν^5 and also to the level density at the end state, which decreases sharply for the transitions of larger energy.

LA - 24 (37)

February 15, 1944

LECTURE SERIES ON NUCLEAR PHYSICS

Sixth Series: Diffusion Theory   Lecturer: R.F. Christy

LECTURE 37: DIFFUSION EQUATION AND POINT SOURCE SOLUTIONS

In the first lectures we will discuss the diffusion of neutrons which have come into energy equilibrium with their surroundings (thermal neutrons). Later we will discuss the slowing down and diffusion of fast neutrons and corrections to the diffusion theory of thermal neutrons.

The Diffusion Equation

The flux of thermal neutrons is $-D\nabla n$ where n is the thermal neutron density and D the diffusion constant. D can be obtained from simple kinetic theory arguments and equals $\lambda v/3$ where λ is the mean free path and v the neutron velocity. The "standard" velocity of thermal neutrons is 2.2 Km/sec. λ is related to the total cross-section σ by the relation $\lambda = 1/N\sigma$ where N is the number of nuclei per cc.

The diffusion equation can then be written

$$\text{div (D grad n)} + \text{production/cc sec.} - \text{absorption/cc sec.} = \partial n/\partial t$$

The production/cc sec. is usually denoted by q, a function of position. If σ_a is the absorption cross-section, the absorption/cc sec. is equal to $N\sigma_a n v$; or, writing $\Lambda = 1/N\sigma_a$ = the mean free path for absorption, the absorption/cc sec equals $(v/\Lambda)n = n/\tau$ where τ is the mean life of the neutrons.

This leads to the diffusion equation in the following forms.

$$D\nabla^2 n + q - \frac{n}{\tau} = \frac{\partial n}{\partial t}$$

$$\frac{\lambda v}{3}\nabla^2 n + q - \frac{v}{\Lambda} n = \frac{\partial n}{\partial t}$$

If we consider the time independent diffusion equation ($\frac{\partial n}{\partial t} = 0$) we get

$$\frac{\lambda v}{3} \nabla^2 n - \frac{v}{\Lambda} n + q = 0$$

$$D\nabla^2 n - \frac{n}{\tau} + q = 0$$

and $$\nabla^2 n - \frac{n}{L^2} + \frac{3}{\lambda v} q = 0$$

where $L^2 = \lambda\Lambda/3 = D\tau$. L is called the diffusion length.

<u>Plane Source in an Infinite Medium</u>

Let the source be of strength Qn/cm^2 and occupy the plane z = 0. With q = 0 everywhere except z = 0, the equation reads

$$\frac{d^2 n}{dz^2} - \frac{n}{L^2} = 0$$

which has solutions $e^{z/L}$ and $e^{-z/L}$. Since the neutron density must approach zero for $z = \pm\infty$ the correct solution must be $n = A e^{-|z|/L}$.

In order to relate A to Q, we note that the flow out from the source per cm^2 is

$$-\frac{2\lambda v}{3} \left.\frac{dn}{dz}\right)_{z=0} = \frac{2\lambda v}{3} \frac{A}{L} = Q$$

$$\text{so } A = \frac{3LQ}{2\lambda v}$$

$$\text{and } n = \frac{3LQ}{2\lambda v} e^{-|z|/L}$$

This equation can be used to determine L experimentally for such materials as paraffin or water where $L \approx 3$ cm is small compared to convenient transverse dimensions of about 1 meter. A source of thermal neutrons is not directly available but can be obtained as follows. A fast neutron source is placed below the plane $z = 0$ in some slowing medium. At $z = 0$ a removable sheet of Cd (which selectively absorbs thermal neutrons) is placed. Above $z = 0$, data on n is obtained for the Cd in place and with no Cd. The difference represents the effect of the Cd which is a thermal source or sink and follows the above equation.

If the diffusing medium cannot be considered infinite in the transverse direction and if the source can be considered to have the transverse distribution $\cos \pi x/a \;\; \cos \pi y/a$ where a is the transverse dimension, then the solution is

$$n = A \cos \frac{\pi x}{a} \quad \cos \frac{\pi y}{a} \quad e^{-z/b}$$

Substitution in the diffusion equation

$$\frac{d^2n}{dx^2} + \frac{d^2n}{dy^2} + \frac{d^2n}{dz^2} - \frac{n}{L^2} = 0$$

leads to

$$-\frac{\pi^2}{a^2} - \frac{\pi^2}{a^2} + \frac{1}{b^2} - \frac{1}{L^2} = 0$$

$$\text{or} \quad \frac{1}{b^2} = \frac{1}{L^2} + \frac{2\;{}^{2}}{a^2}$$

which determines b. This relation is frequently used for the determination of L in media where L is relatively long, of order 1 ft.

Point Source in an Infinite Medium

Here we write the diffusion equation with $q = 0$ in spherical coordinates, considering only spherically symmetric solutions.

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dn}{dr}\right) - \frac{n}{L^2} = 0$$

if $u = rn$ then u satisfies the equation

$$\frac{d^2u}{dr^2} - \frac{u}{L^2} = 0$$

The condition that n be finite at $r = \infty$ gives

$$u = A_0 e^{-r/L}$$

and $$n = \frac{A}{r} e^{-r/L}$$

The source strength

$$Q = \frac{\lambda v}{3} 4\pi r^2 \left.\frac{\partial n}{\partial r}\right)_{r=0} = 4\pi A \frac{\lambda v}{3}$$

so $$n = \frac{3Q}{4\pi\lambda v} \frac{1}{r} e^{-r/L}$$

Depression of Neutron Density Near a Foil

A simple expression for the depression of the neutron density near a foil can be obtained only if we assume the foil to have the form of a spherical shell. Let its thickness be t, absorption cross-section σ_a, and its nuclear density be N.

The diffusion equation is

$$D\nabla^2 n - \frac{n}{\tau} + q = 0$$

Let $n = q\tau - \frac{A}{r} e^{-r/L}$ which satisfies the equation and has in general the right form. The number of neutrons absorbed per second by a foil of radius r_o is

$$N\sigma_a \, t n(r_o) v 4\pi r^2 = \frac{\lambda v}{3} 4\pi r^2 \left.\frac{\partial n}{\partial r}\right)_{r_o}$$

$$\text{or}\quad \frac{3N\sigma_a t}{\lambda}\left(q\tau - \frac{A}{r_o}\; e^{-r_o/L}\right) = \frac{A}{r_o} e^{-r_o/L}\left(\frac{1}{r_o} + \frac{1}{L}\right)$$

$$\frac{A}{r_o}\; e^{-r_o/L}\left\{\frac{1}{r_o} + \frac{1}{L} + \frac{3N\sigma_a t}{\lambda}\right\} = \frac{3N\sigma_a t}{\lambda}\; q\tau$$

$$\frac{A}{r_o}\; e^{-r_o/L} = \frac{q\tau}{1 + \dfrac{\lambda/r_o + \lambda/L}{3N\sigma_a t}}$$

The fractional depression in neutron density at the foil is

$$\frac{\frac{A}{r_o}\; e^{-r/L}}{q\tau} = \frac{1}{1 + \dfrac{\lambda/r_o + \lambda/L}{3N\sigma_a t}}$$

which can be considerably larger than the fraction of neutrons absorbed in a single traversal of the foil which is $N\sigma_a t$.

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February 17, 1944

LECTURE SERIES ON NUCLEAR PHYSICS

Sixth Series: Diffusion Theory Lecturer: R.F. Christy

LECTURE 38: POINT SOURCE PROBLEMS AND ALBEDO

Effect on a Foil on Neutron Density

In the last lecture we considered the effect of a detector, such as an indium foil, on the neutron density in a medium in which thermal neutrons were diffusing. It was shown how, as a first approximation, the foil could be replaced by a sphere and fairly simple results obtained. Let us now consider a more exact approach to the problem.

Suppose we consider the element of area (dA) of the foil surface as a point absorber. If p is the coordinate of the element dA and r is the coordinate of the point at which we want to know the neutron density, then the effect of a point sinh such as dA is given by:

$$\frac{Ce^{-\frac{|r-p|}{L}}}{|r - p|}$$

To evaluate C we consider the following:

The number of neutrons absorbed per second may be set equal to the number passing through the surface of a very small sphere surrounding our point sinh. Now the number of neutrons absorbed per second equals $Nt\sigma_a nvdA$ where:

N = number of atoms/cc in foil

t = thickness of foil

σ_a = absorption cross-section of foil material

v = neutron velocity

The number of neutrons passing into a very, very small sphere is given by:

$$\frac{\lambda v}{3} \lim_{r \to o} 4\pi r^2 \frac{d}{dr}\left(\frac{Ce^{-r/L}}{r}\right) = -\frac{4\pi\lambda v}{3} C$$

Therefore:

$$\frac{4\pi\lambda v}{3} C = -Nt\sigma_a n \, v \, dA$$

$$C = -\frac{3Nt\sigma_a nv}{4\pi\lambda v} dA = -\frac{3Nt\sigma_a n}{4\pi\lambda} dA$$

The effect of the whole foil is then the sum of the effects of each element. On adding these effects and passing to the limit as the number of subdivisions is increased we get the total effect of the foil at any point r to be

$$-\frac{3Nt\sigma_a}{4\pi\lambda} \int_{foil} n(p)\frac{e^{-|r-p|}}{|r-p|} dA$$

If we assume our source to be such that in the absence of the foil the neutron density in the region in which we are interested is everywhere constant and equal to one, then the neutron density at any point is merely given by the sum of the contributions of the foil and the source without the foil

Thus

$$n(r) = 1 - \frac{3Nt\sigma_a}{4\pi\lambda} \ {}_{foil} \ n(p) \ \frac{e^{-|r-p|}}{|r-p|} dA$$

The more precise approach to our problem therefore is seen to lead to an integral equation. Since this equation is difficult, if not impossible, to solve exactly the usual procedure is to have recourse to approximating the foil by a sphere.

Point Source in a Finite Sphere

As another example of the way in which the diffusion equation may be applied let us consider the problem of a point source in a finite spherical medium. The general solution in such a case is:

$$n = \frac{A}{r} e^{-r/L} + \frac{B}{r} e^{r/L}$$

where A and B are arbitrary constants to be determined. B may be found in terms of A by means of the relation; $n(R) = 0$ where R is the outer radius of the sphere.

$$n(R) = 0 = \frac{A}{R} e^{-R/L} + \frac{B}{R} e^{R/L}$$

$$\therefore B = -A\, e^{-2R/L} \quad \text{and the solution becomes:}$$

$$n = \frac{A}{r}\left(e^{-r/L} - e^{+(r-2R)/L}\right) = \frac{2A}{r} e^{-R/L} \sinh \frac{R-r}{L}$$

The constant A may then be determined by equating the number of neutrons passing through a very small sphere to the number produced per second by the source.

Point Source in an Infinitely Long Column

Up until now the solutions to the diffusion equation that we have found have involved only a few terms. In general such representations will not satisfy the boundary conditions. Instead we must have recourse to more complex methods, (e.g. the use of Fourier Series). As an illustration of such a problem we can take the case of a point source embedded in a rectangular medium of length a in both x and y directions but infinite in extent in the + and - a directions. We take axes such that the source which is at the center of the medium is at the point (o, o, o). The general solution of the equation $\nabla^2 n - \frac{n}{L^2} = 0$ is:

$$n = \sum_{mn} A_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{a} e^{-|a|/b_{mn}}$$

To determine b_{mn} we can use the fact that each term separately must satisfy the differential equation. Hence b_{mn} is given by the relation:

$$-\frac{n^2\pi^2}{a^2} - \frac{m^2\pi^2}{a^2} + \frac{1}{b^2_{mn}} = \frac{1}{B^2}$$

To find the coefficients A_{mn} we consider the fact that we have a point source. Such a source can be represented as $\delta(x,y)$ where δ stands for the Dirac delta function. Here we are assuming a source of slow neutrons, though in practice a fast neutron source is used. Let us expand $\delta(x,y)$ in a Fourier series.

$$\text{i.e. } \delta(x,y) = \sum_{mn} B_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{a}$$

We determine the B_{mn}'s as follows:

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy\, \delta(x,y) \cos\frac{m\pi x}{a} \cos\frac{m\pi y}{a} = 1.$$ *by the definition of the delta functions*

$$B_{mn} \int_{-\frac{a}{2}}^{\frac{a}{2}} \cos^2\frac{m\pi x}{a}\, dx \int_{-\frac{a}{2}}^{\frac{a}{2}} \cos^2\frac{n\pi y}{a}\, dy = \frac{a^2}{4} B_{mn} = 1$$

$$\therefore B_{mn} = \frac{4}{a^2}$$

To relate the A's and B's we must use the fact that the number of neutrons flowing out of the plane z = 0 per second in a given mode must equal the number produced per second in that mode. The latter is equal to B_{mn} .

$$\text{Number flowing out} = \frac{2\lambda v}{3} \frac{\partial B_{mn}}{\partial z}\Big]_{z=0} = \frac{2\lambda v}{3b_{mn}} A_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{a}$$

$$\therefore B_{mn} = \frac{4}{a^2} = \frac{2\lambda v}{3b_{mn}} A_{mn}$$

$$A_{mn} = \frac{6b_{mn}}{\lambda v a^2}$$

Hence our solution is:

$$n = \sum_{mn} \frac{6b_{mn}}{\lambda v a^2} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{a}\, e^{-|z|/b_{mn}}$$

If the source strength were Q instead of 1 the above solution must be multiplied by a factor of Q. We note that B_{mn} decreases rapidly with increasing m and n and so at great distances the only significant term in our series will be the first. Thus at great distances from our source:

$$n = \frac{6b_{11}}{\lambda v a^2} Q \cos\frac{\pi x}{a} \cos\frac{\pi y}{a}\, e^{-|z|/b_{11}}$$

This presents a very convenient method of measuring L since the decrease of n with z enables us to find b_{11} and from b_{11} we can readily compute L.

<u>Solution of the Diffusion Equation in an Infinite Medium with q as a Function of Position.</u>

As another type problem suppose we consider the solution of the diffusion equation with a term due to the production of neutrons (q) by a rather special method. The equation involved is:

$$\frac{\lambda v}{3} \nabla^2 n - \frac{vn}{\Delta} + q = 0$$

If $q = 0$, the solution is (assuming an infinite medium spherical symmetry)

$$n = \frac{A}{r} e^{-r/L} \quad \text{, where A is some constant.}$$

As shown in a previous lecture A can be related to the source strength Q by:

$$A = \frac{3Q}{4\pi \lambda v}$$

Hence:

$$n = \frac{3Q}{4\pi\lambda v} \quad \frac{e^{-r/L}}{r}$$

Suppose now $q = f(x, y, z)$.

We can solve the problem as follows: The solution for a point source has the form given above. Our function q can then be treated as an infinite number of point sources continuously distributed throughout the medium. The total effect will then be the sum of the effects due to each point. If r is the point at which we want to know the neutron density and r' the point where our source is located we have:

$$n(r) = \int_{\text{vol.}} \frac{3q(r')}{4\pi\lambda v} \frac{1}{|r - r'|} e^{-|r-r'|} d\,\text{vol}'$$

Sometimes the above integral can be readily evaluated and under such conditions represents a very great simplification of the problem. In utilizing this method it is very important to take advantage of whatever symmetry the problem offers. Thus if plane symmetry were exhibited the appropriate plane solution of the diffusion equation should be chosen. If cylindrical symmetry were the case it would be well to find the solution for such conditions and then apply the method here followed of integrating to get the contributions of the many point sources making up the q function.

Albedo

A quantity of considerable utility in neutron diffusion is the albedo or reflection coefficient. If a finite medium with a point source were left in empty space, the neutrons reaching the bounding surface would diffuse out and be lost. Now, if another medium were surrounding the first one, some of the neutrons reaching the boundary would be reflected back in, thus increasing the neutron density in our original medium. The magnitude of the reflection effect is measured by the albedo γ of the, in this case, outside medium. Actually the albedo of a medium will depend, not only on the medium, but also on the angular distribution of neutrons falling on the medium. We will neglect this effect.

Albedo of Various Media.

For this purpose we must consider the angular distribution of neutrons at a point. We let be the cosine of the angle between the velocity of the neutron and the radius vector and write

$$nv(r\,\mu) = \frac{1}{2}\left[A(r) + \mu B(r)\right]$$

so $$nv(r) = \int_{-1}^{1} nv(r,\mu)\,d\mu = A(r)$$

The net flow of neutrons $= -\frac{\lambda}{3}\frac{d(nv)}{dr} = -\frac{\lambda}{3}A'(r) = \int_{-1}^{1} nv(r\mu)\,\mu\,d\mu = \frac{B(r)}{3}$

$$\text{so} \quad B(r) = -\lambda A'(r)$$

$$\text{and} \quad nv(r,\mu) = \frac{1}{2} A(r) - \mu\lambda A'(r)$$

The total flow in the positive radial direction is called the flow out.

$$\text{flow out} = \int_0^1 nv(r,\mu)\mu d\mu = \frac{1}{4}\left[A(r) - \frac{2}{3}\lambda A'(r)\right]$$

The total flow in the negative radial direction is called the flow in.

$$\gamma = \text{flow in} = \int_0^{-1} nv(r,\mu)\mu d\mu = \frac{1}{4}\left[A(r) + \frac{2}{3}\lambda A'(r)\right]$$

Now the albedo of the inside surface R of the medium is

$$\gamma = \frac{\text{flow in}}{\text{flow out}} = \frac{A(R) + 2/3\,\lambda A'(R)}{A(R) - (2/3)\lambda A'(R)} = \frac{nv(R) + \frac{2}{3}\lambda \left.\frac{d(nv)}{dr}\right)_R}{nv(R) - \frac{2}{3}\lambda \left.\frac{d(nv)}{dr}\right)_R}$$

$$\gamma = \frac{1 + \frac{2}{3}\frac{\lambda}{n}\left.\frac{dn}{dr}\right)_R}{1 - \frac{2}{3}\frac{\lambda}{n}\left.\frac{dn}{dr}\right)_R} \quad \text{or} \quad -\frac{\lambda}{n}\left.\frac{dn}{dr}\right)_R = \frac{3}{2}\,\frac{1-\gamma}{1+\gamma}$$

We can take $n = \frac{1}{r} e^{-r/L}$

$$\text{so} \quad -\frac{dn}{dr} = \frac{1}{r} e^{-r/L} \left(\frac{1}{r} + \frac{1}{L}\right)$$

$$\gamma = \frac{1 + \frac{2}{3}\left(\frac{\lambda}{R} + \frac{\lambda}{L}\right)}{1 + \frac{2}{3}\left(\frac{\lambda}{R} + \frac{\lambda}{L}\right)}$$

The larger is R the better the albedo and for R , the albedo approaches that of a plane

$$\gamma = \frac{1 - \frac{2}{3}\frac{\lambda}{L}}{1 + \frac{2}{3}\frac{\lambda}{L}}$$

If in addition we make the medium non-absorbing, $L = \infty$, $\gamma = 1$. This is actually obvious: any neutron entering a non-absorbing half infinite medium will eventually return.

It is apparent that the formula breaks down when R is small (γ would be negative) compared to λ. This is when diffusion theory breaks down.

Let us look at the inverse problem, the albedo of a sphere. Here

$$n = \frac{1}{r} \sinh \frac{r}{L}$$

$$-\frac{dn}{dr} = +\frac{n}{r} - \frac{\coth (r/L)}{L} n$$

$$\gamma = \frac{\text{flow out}}{\text{flow in}} = \frac{1 - \frac{2}{3}\left(\frac{\lambda}{L} \coth \frac{R}{L} - \frac{\lambda}{R}\right)}{1 + \frac{2}{3}\left(\frac{\lambda}{L} \coth \frac{R}{L} - \frac{\lambda}{R}\right)}$$

For an infinitely long cylindrical hole, the differential equation to be solved is

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dn}{dr}\right) - \frac{n}{L^2} = 0$$

so $n = K_o \left(\frac{r}{L}\right)$ a Bessel function of second kind of imaginary argument. For $\frac{R}{L}$ small we get

$$\gamma = \frac{1 - \frac{2\lambda}{3R} \quad \frac{1}{\ln (2L/CR)}}{1 + \frac{2\lambda}{3R} \quad \frac{1}{\ln (2L/CR)}}$$

$\ln C$ = Euler's constant

This formula shows a surprising result that for $L = \infty$ (no absorption) $\gamma = 1$.

<u>Boundary conditions expressed in terms of Albedo</u>

The boundary conditions at the interface of two media are, in diffusion theory $n_1 = n_2$ at the boundary

$$\text{and } \frac{\lambda_1 v}{3} \frac{dn_1}{dr} = \frac{\lambda_2 v}{3} \frac{dn_2}{dr} \text{ at the boundary}$$

Instead of one of these we can use the ratio

$$\left.\frac{\lambda_1 (dn_1/dr)}{n_1}\right)_B = \left.\frac{\lambda_2 (dn_2/dr)}{n_2}\right)_B$$

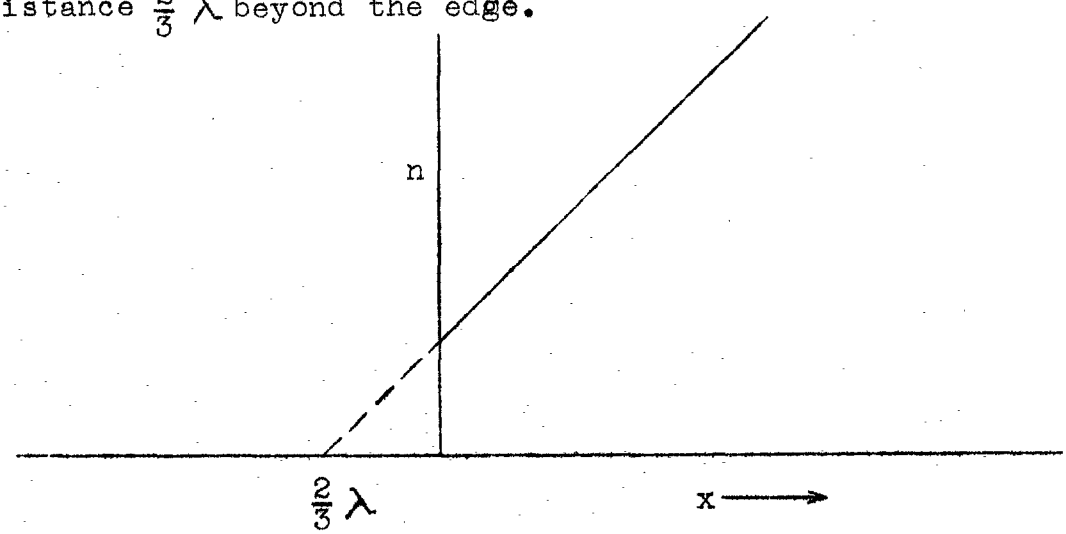
Now this ratio for a medium can be written in terms of the albedo so

$$\left.-\frac{\lambda_1 (dn_1/dr)}{n_1}\right)_B = \frac{3}{2} \frac{1 - \gamma_2}{1 + \gamma_2}$$

is another possible form of the boundary condition.

Extrapolated End Point

Let us examine the neutron density in a plane semi-infinite non-absorbing slab. We will show that if the neutron density inside the slab is continued to the edge and beyond it would vanish at a distance $\frac{2}{3}\lambda$ beyond the edge.



We have, as before

$$nv(x, \mu) = \frac{1}{2} (A(x) - \mu\lambda A'(x))$$

The solution deep in the slab will be $nv = A(x) = a(1 + \frac{x}{L})$ where L is the extrapolated end point. Since $A(x) = 0$ for $x = -L$

$$A'(x) = \frac{a}{L}$$

$$nv(x,\mu) = \frac{a}{2}\left[1 + \frac{x}{L} - \mu\frac{\lambda}{L}\right]$$

The total flow in the $+x$ direction is

$$\frac{a}{4}\left[1 + \frac{x}{L} - \frac{2}{3}\frac{\lambda}{L}\right]$$

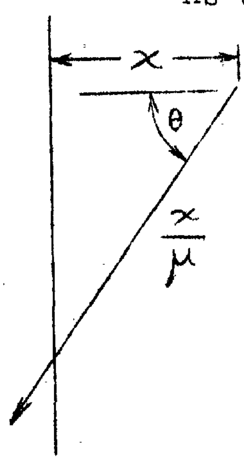
which must vanish at $x = 0$ since there are no neutrons entering the slab.

$$L = \frac{2}{3}\lambda$$

An exact solution of this problem leads to $L = .7104\lambda$ actually.

Angular Distribution of neutrons Emerging from a Surface

As above we may take $nv(x) = a(1 + \frac{3}{2}\frac{x}{\lambda})$



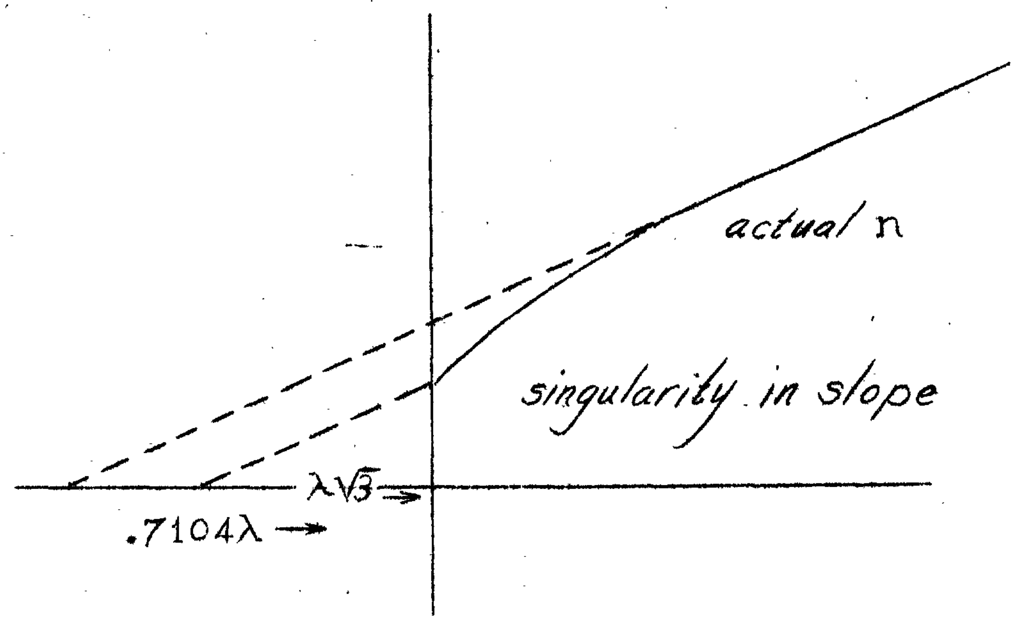
The $\frac{\text{collisions}}{\text{see cm}^2}$ at x are $\frac{adx}{\lambda}\left(1 + \frac{3}{2}\frac{x}{\lambda}\right)$

We may consider these neutrons uniformly distributed in angle and calculate the probability of escape from the surface at angle μ (per unit range of μ)

$$nv(\mu) = \int_0^\infty \frac{a}{2\lambda}\left(1 + \frac{3}{2}\frac{x}{\lambda}\right) e^{-\frac{x}{\mu\lambda}}dx$$

$$= \frac{a}{2}\left(\mu + \frac{3}{2}\mu^2\right)$$

So then the flux from the surface $\sim \cos\theta + \frac{3}{2}\cos^2\theta$



Actual Distribution Near a Plane Boundary of a Non-absorbing Medium

Actual angular distribution in $n \sim \mu + \sqrt{3}\,\mu^2 \ldots\ldots\ldots$

The extrapolation of the asymptotic solution vanishes at .7104 λ from the surface. Very close (in a mean free path) to the surface the neutron density decreases and has a singularity in slope at the surface. The value of n at the surface is $\frac{1}{.7104\sqrt{3}}$ of the value of the straight line solution at the surface.

LA-24 (39)

February 24, 1944

LECTURE SERIES ON NUCLEAR PHYSICS

Sixth Series: Diffusion Theory Lecturer: R. F. Christy

LECTURE 39: THE SLOWING DOWN OF NEUTRONS

Fermi's Age Equation

The preceding lectures have all dealt with the diffusion of thermal neutrons. Let us now consider how neutrons coming from a fast neutron source are slowed down. In the discussion the quantity q will be very convenient. It is defined as follows:

q = No. of neutrons crossing energy E per cc per sec

It is obvious that the total no. of neutrons crossing energy E must equal the no. produced per second. Hence from our definition of q we must have:

$$\int_{vol} q(r,E)d\ vol = Q$$

where Q is the total number of neutrons produced per second.

The mechanism by which the neutrons are slowed down is that of collision with other particles. The amount of kinetic energy lost by a neutron in a given encounter may be computed by the methods of classical mechanics as applied to elastic collisions. In general, if a neutron hits a particle of mass M the ratio of its original energy (E) to its final energy (E') is a function both of the mass M and the angle between the original path of the neutron and its path after the collision. By appropriate averaging over all angles we can conclude that

$$\left(\frac{E'}{E}\right)_{av} = C(M)$$

where C is a constant which depends only on M. This relation is usually written as:

$$\Delta \ln E = \ln \frac{E}{E'} = \xi_M$$

where ξ is the average of $\ln \frac{E}{E'}$.

It can be shown that:

$$\xi_M = 1 - \frac{(M-1)^2}{2M} \ln \frac{M+1}{M-1}$$

$$\xi_M = \frac{2}{M+1}$$

where the approximation is better for heavier nuclei (i.e. for larger M). A few sample values are:

for hydrogen: $\xi_H = 1$

for deuterium: $\xi_D = 0.725$

for carbon: $\xi_C = 0.158$

for oxygen: $\xi_O = 0.120$

Using the above we may find the mean number of collisions suffered by a neutron in slowing down. Suppose the neutrons start off with energies of 2 Mev and end up at thermal energies (about 1/30 ev). Then the mean number of collisions:

$$\frac{\ln (2 \times 10^6 \times 30)}{\xi} = \frac{18}{\xi}$$

The treatment now to be presented for slowing down is somewhat limited in its application - even more so than is ordinary diffusion theory. In the latter case our only important approximation was that λ grad n be small. In slowing down theory it will be demanded that the number of collisions be large and also that the mean free path (λ) not vary much between collisions. These assumptions are necessary if we are to be able to replace the essentially discontinuous process of energy loss by collision by a continuous process. (It may be noted that the most common slowing down substances, such as paraffin and water, fail to satisfy the conditions very well and so any treatment of paraffin, water, and similar materials by the methods given here may result in considerable error.)

The starting point in treating slowing down phenomena is the so-called Fermi Age Equation. We shall give here a non-rigorous justification rather than a formal derivation of the equation. Our standard diffusion equation was:

$$\frac{\lambda v}{3} \nabla^2 n = \frac{\delta n}{\delta t}$$

By analogy we write:

$$\frac{\Delta v}{3} \nabla^2 q = \frac{\delta q}{\delta t}$$

If we consider t in this last equation to be a variable depending on the past history of the neutrons we can relate this to the neutron energy. Thus: $\frac{d(\ln E)}{dt} = \Delta \ln E$ per collision λ number of collisions per second.

But $\Delta \ln E$ per collision $= \xi$

and number of collisions per second $= \frac{v}{\lambda}$

$$\therefore \frac{\Delta(\ln E)}{dt} = \xi \frac{v}{\lambda}$$

Hence: $$dt = \frac{\lambda\, d(\ln E)}{\xi v}$$

Then our original equation becomes:

$$\nabla^2 q = \frac{\delta q}{\frac{\lambda^2}{3}\,\delta \ln E}$$

We introduce the quantity τ (the "Age") by the relation:

$$d\tau = \frac{\lambda^2}{3\xi}\, d \ln E$$

On substituting we get:

$$\nabla^2 q = \frac{\delta q}{\delta \tau} \qquad \text{(Fermi's Age Equation)}$$

We note that the "Age" has the dimensions of length squared. It plays very much the same role in slowing down as the quantity L^2 did in thermal neutron diffusion. τ is a function of the energy (E) to which the neutrons have been slowed and may be found from the relation:

$$\tau = \int_E^{E_0} \frac{\lambda^2}{3\xi}\, d(\ln E)$$

where E_0 is the energy with which the neutrons leave the source.

Point Source Solution

As our first example of the use of the Age Equation let us calculate the distribution of neutrons in space and energy coming from a point source. We shall consider an infinite medium with spherical symmetry. Our equations are:

$$\tau = \int_{E}^{E_o} \frac{\lambda^2}{3\xi} \frac{dE}{E} \; : \; \nabla^2 g = \frac{\delta g}{\delta \tau}$$

The equation involving the Laplacian reduces to:

$$\frac{\delta^2 (rg)}{\delta r^2} = \frac{\delta^2 (rg)}{\delta \tau}$$

We shall solve this equation by the standard method of separation of variables:

Let $\quad rq = f(r)\, \emptyset\, (\tau)$

Then $\quad \emptyset \dfrac{d^2 f}{dr^2} = f \dfrac{d\emptyset}{d\tau}$

$$\frac{1}{f} \cdot \frac{d^2 f}{dr^2} = \frac{1}{\emptyset} \cdot \frac{d\emptyset}{d\tau}$$

Since the left hand side is a function of r alone and the right of τ alone we must have:

$$\frac{1}{f} \cdot \frac{d^2 f}{dr^2} = \frac{1}{\emptyset} \cdot \frac{d\emptyset}{d\tau} = -k^2$$

Thus we get the two equations:

(1) $\quad \dfrac{d^2 f}{dr^2} + k^2 f = 0$

(2) $\quad \dfrac{d\emptyset}{d\tau} + k^2 \emptyset = 0$

The solution of (1) is: $f = A \sin kr + B \cos kr$

The solution of (2) is: $\emptyset = e^{-k^2\tau}$

$$\therefore \text{ Solution is: } q = \frac{1}{r} (A \sin kr + B \cos kr) e^{-k^2\tau}$$

where A and B are arbitrary constants depending on k. Since q must stay finite at $r = 0$, we must have $B = 0$.

$$\text{Hence: } q = \frac{A}{r} \sin kr \, e^{-k^2\tau}$$

Since we have no boundary conditions to be satisfied, any real value of k will satisfy our equations. The general solution, which is the sum of all possible solutions, is then found by by multiplying the above expression by dk and integrating from $-\infty$ to $+\infty$.

$$\text{i.e. } q = \frac{1}{r} \int_{-\infty}^{\infty} A(k) \sin kr \, e^{-k^2\tau} dk$$

Using the fact that we have a point source we can determine the function A(k). We know that:

$$q(\tau = 0) = C\,\delta(r)$$

where C is a constant to be determined later which depends only on the source strength Q. $\delta(r)$ is a delta function such that:

$$\delta(r) = 0,\ r \neq 0 \text{ and } \int_0^{\infty} r^2 \delta(r) dr = 1$$

Substituting in the above we get:

$$r\,C\,\delta(r) = \int_{-\infty}^{\infty} A(k) \sin kr \, dk$$

As the integrand is symmetric (as will be seen from later results) written as:

$$rC\delta(r) = 2\int_0^\infty A(k)\sin kr\, dk$$

Multiplying by $\frac{1}{\sqrt{2\pi}}$ yields

$$\frac{rC\delta(r)}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}\int_0^\infty A(k)\sin kr\, dk$$

Now the Fourier Sine Transform tells us that

$$\text{If} \quad f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \phi(u)\sin xu\, du$$

$$\text{then:} \quad \phi(u) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin ux\, dx$$

Applying this to our problem:

$$A(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{C}{\sqrt{2\pi}}\, r\delta(r)\sin kr\, dr$$

$$= \frac{C}{\pi}\int_0^\infty r^2\delta(r)\frac{\sin kr}{r}\, dr$$

By the definition of the delta function this becomes

$$A(k) = \frac{C}{\pi}\lim_{r\to 0}\ \sin\frac{kr}{r} = \frac{kC}{\pi}$$

Hence

$$g = \frac{C}{\pi r}\int_{-\infty}^{\infty} k\sin kr\, e^{-k^2\tau}\, dk$$

Substituting $\frac{e^{ikr} - e^{-ikr}}{2i} = \sin kr$ and then multiplying and

dividing by $e^{r^2/4\tau}$ yields

$$g\,\frac{Ce^{-r^2/4\tau}}{2\pi i r}\int_{-\infty}^{\infty}\left(e^{-\tau(k - ir/2\tau)^2} - e^{-\tau(k + ir/2\tau)^2}\right)k\, dk$$

To evaluate the first integral let

$$z = (k - \frac{ir}{2\tau})$$

Then:

$$-\int_{-\infty}^{\infty} e^{-\tau(k-ir/2\tau)^2} k\,dk = \int_{-\infty}^{\infty} e^{-\tau z^2} z\,dz + \frac{ir}{2\tau}\int_{-\infty}^{\infty} e^{-\tau z^2} dz$$

The first term on the right yields zero while the second becomes:

$$\frac{ir}{2\tau^{3/2}}\sqrt{\pi}$$

Similarly

$$-\int_{-\infty}^{\infty} e^{-\tau(k+ir/2\tau)^2} k\,dk = \frac{ir}{2\tau^{3/2}}\sqrt{\pi}$$

Thus:

$$g = \frac{Ce^{-r^2/4\tau}}{2\pi^{1/2}\tau^{3/2}}$$

To find C we remember that:

$$\int_{vol} q \; d \; vol = Q$$

$$\therefore C\int_{-\infty}^{\infty} r^2 d^{-r^2/4\tau} dr = \frac{\tau^{3/2} Q}{2\pi^{1/2}}$$

Let $u = r^2/4\tau$ then:

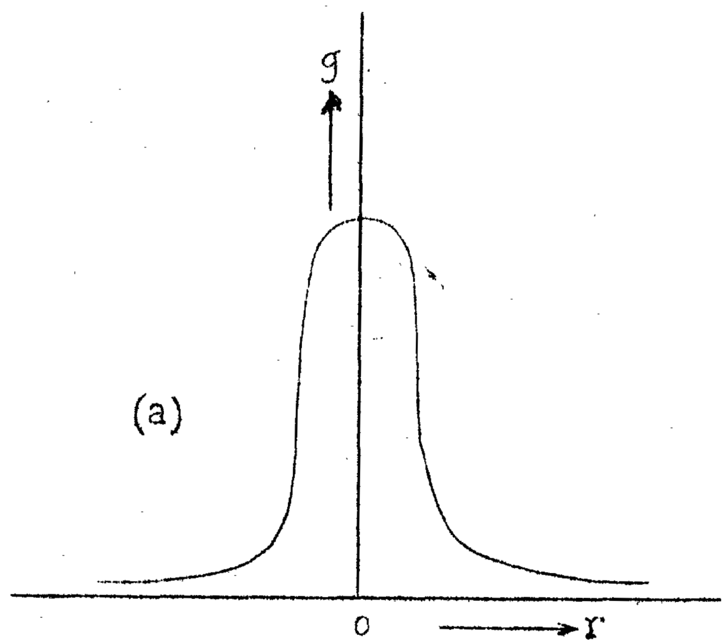
$$\int_0^{\infty} r^2 e^{-r^2/4\tau} dr = 4\tau^{3/2}\int_0^{\infty} u^{1/2} e^{-u} du = 2\pi^{1/2}\tau^{3/2}$$

$$\therefore C = Q/4\pi$$

Hence:

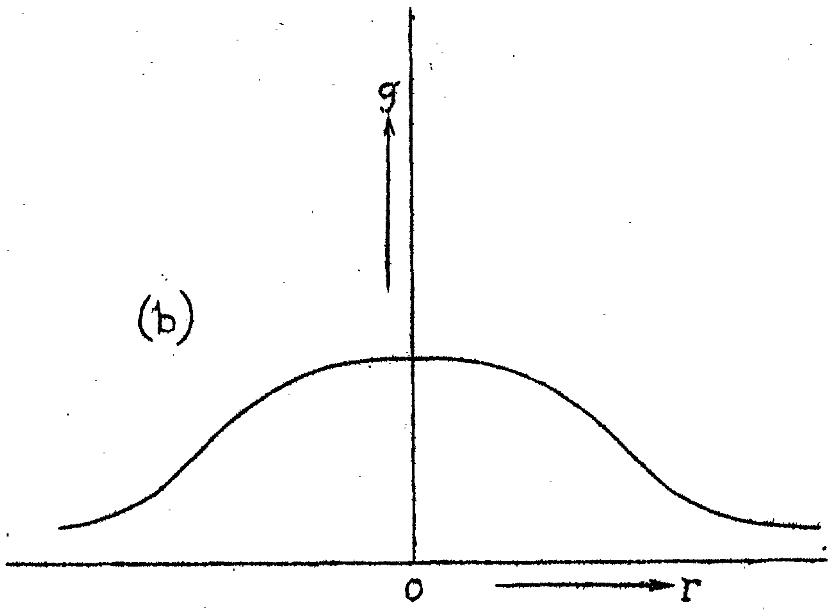
$$q = \frac{Q e^{-r^2/4\tau}}{(4\pi\tau)^{3/2}}$$

That this result is what we should expect is easily seen from the following argument. Since at $r = 2\sqrt{\tau}$ q has reduced to 1/e of its value at the origin, we see that τ is a measure of the width of curve representing q. If τ were very small our curve should be very narrow (a).



This is physically plausible, since small τ in dicates fast neutrons and these we would expect to be grouped near the source.

On the other hand large τ means slower neutrons. These would be expected to be more uniformly distributed than the faster ones. That is just what our formula leads us to conclude.



<u>Plane Source Representations.</u>

If instead of a point source we have an infinite plane source the solution would be of the form:

$$q = Q\,\frac{e^{-z^2/4\tau}}{\sqrt{4\pi\tau}} \qquad Q = \text{neuts/cm}^2 \text{ sec from source}$$

Let us suppose now that we have a medium finite and of width (a) in both x and y directions and infinite in the - and - z directions. Assume a point source at (0, 0, 0). A solution of the equation:

$$\nabla^2 q = \frac{\delta q}{\delta \tau}$$

which satisfies the boundary conditions at $x = \pm a/3$, $y = \pm a/2$ is:

$$q \quad C(k)A_{\ell,m} \cos \frac{\ell\pi x}{a} \cos \frac{m\pi x}{a} \cos kze^{-\rho^2\tau}$$

where ℓ and n are positive integers, C(k) is a constant depending on k, and

$$p^2 = \frac{(\ell^2 + m^2)}{a^2}\pi^2 + k^2. \qquad \text{(k is any real number)}$$

The general solution is then found by adding up all possible solutions. This means summing over m and ℓ and integrating over k. Thus

$$q = \int_{-\infty}^{\infty} C(k) \sum_{m,\ell} A_{\ell,m} \cos\frac{\ell\pi x}{a} \cos\frac{\ell\pi y}{a} \cos kz \, e^{-p^2\tau} dk$$

Applying the orthogonality principle, integrating, and using the fact that a point source may be represented by a delta function we get:

$$q = \frac{16Q}{a^2\sqrt{\pi\tau}} \sum_{\ell,m} e^{-(\ell^2+m^2)\pi^2\tau/a^2} \cos\frac{\ell\pi x}{a} \cos\frac{\ell\pi y}{a} e^{-z^2/4\tau}$$

We note that for $\tau \gg a$ (i.e. for the lower energy neutron) the higher harmonics die out. Hence, with this approximation our expression becomes:

$$q = \frac{16Q}{a^2\sqrt{\pi\tau}} e^{-2\pi^2\tau/a^2} \cos\frac{\pi x}{a} \cos\frac{\pi y}{a} e^{-z^2/4\tau}$$

LA-24 (40)

February 29th, 1944

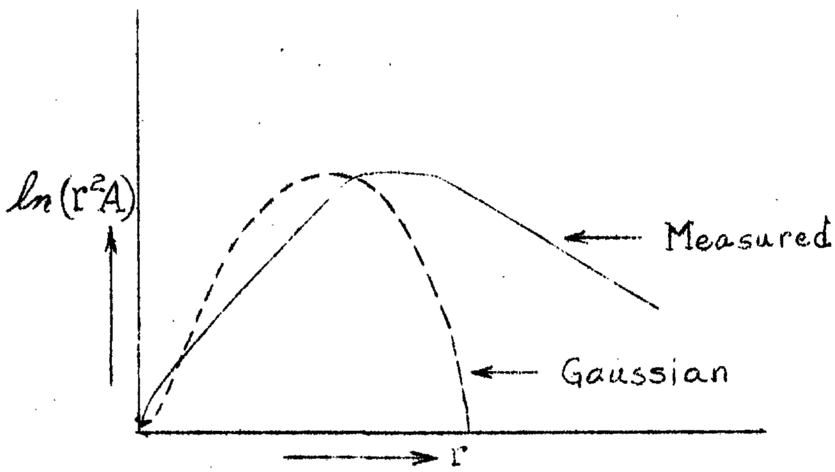
LECTURE SERIES ON NUCLEAR PHYSICS

Sixth Series: Diffusion Theory Lecturer: R. F. Christy

LECTURE V: THE SLOWING DOWN OF NEUTRONS IN WATER

In this lecture we shall consider the following question. What happens when a Ra-Be source of neutrons is placed in a large water tank? What is the distribution of neutrons in space and energy?

Suppose we were to measure the distribution by means of a cadmium covered indium foil. Such a counter records the number of neutrons of the indium resonance energy (slightly above 1 ev). Plotting $\ln(r^2A)$ against r (where r is the distance from the source and A is the activity measured) we obtain results such as those in the diagram.



We note that at large values of r the curve is practically straight. This implies that in that region:

$$A \sim \frac{1}{r^2} e^{-r/\lambda}$$

We also note that the curve of measured neutron density differs considerably at large r from what it would look like if it followed a Gaussian distribution (dotted line).

The above distribution of neutrons is readily recognized as being the same as that of neutrons from a point source which have not yet had any collisions. That this is so is seen from the fact that the probability that a neutron will go a distance r without a collision is $e^{-r/\lambda}$ and that we must have a factor of $1/4\pi r^2$ to satisfy geometrical considerations.

Now it is easy to demonstrate using classical mechanics that in an elastic collision with a proton a neutron can never be scattered at an angle greater than 90°. Hence the average scattering angle will be between 0° and 90°. We see, therefore, that on colliding with protons the direction of the neutrons is not much changed and so we would expect the neutron density to follow a law similar to the above (i.e. we expect $A \sim (1/r^2)e^{-r/\lambda}$ where λ is actually somewhat greater than the true mean (free path). A further reason why we should expect such a distribution is seen when we remember that the scattering cross-section for hydrogen decreases rapidly with increasing neutron energy. Thus the mean free path is largest at higher energies and so most of the distance traveled by the neutrons is covered in the first few collisions. But in the first few collisions the $(1/r^2)e^{-r/\lambda}$ law is followed. We conclude from this argument also that the last is the law we should expect to find for the neutron density for large r.

A concept which we shall find useful in considering our problem is that of the mean square distance $\overline{r^2}$ traveled by a neutron. This quantity is defined by:

$$\overline{r^2} = \frac{\int_0^\infty r^4 f(r)\,dr}{\int_0^\infty r^2 f(r)\,dr}$$

where f(r) is the functional dependence of the neutron density on the distance (r) from the source.

As an example let us calculate $\overline{r^2}$ for the case where our neutron density follows the $(1/r^2)e^{-r/\lambda}$ law to the first collision and then follows an age diffusion law (i.e. proportional to $e^{-r^2/4\tau}$). Let us denote the first portion by the subscript (1) and the second by the subscript (2). Then:

$$\overline{r^2} = \overline{r^2_1} + \overline{r^2_2}$$

$$\overline{r^2_1} = \frac{\int_0^\infty r^2 e^{-r/\lambda}\,dr}{\int_0^\infty e^{-r/\lambda}\,dr} = \lambda^2\,\Gamma(3) = 2\lambda^2$$

$$r^2_2 = \frac{\int_0^\infty r^4 e^{-r^2/4\tau}\,dr}{\int_0^\infty r^2 e^{-r^2/4\tau}\,dr}$$

Substituting

$$x = \frac{r^2}{4\tau}$$

we get:

$$\overline{r^2_2} = \frac{16\tau^{3/2}\int_0^\infty x^{3/2}e^{-x}dx}{4\tau^{3/2}\int_0^\infty x^{1/2}e^{-x}dx} = 4\tau\frac{\Gamma(5/2)}{\Gamma(3/2)} = 6\tau$$

Hence:

$$\overline{r^2} = \underbrace{2\lambda^2}_{\text{pre-age diffusion}} + \underbrace{6\tau}_{\text{age diffusion}}$$

For water we know $\overline{r^2} = 280$ cm for a Ra-Be source. Furthermore, from our observed curve for $\ln(r^2A)$ we can get λ. It turns out that the $2\lambda^2 \approx 180$. Therefore

$$\tau \approx \frac{280 - 180}{6} \approx 16\frac{2}{3}\ \text{cm}^2$$

In order to calculate the distribution of neutrons let us consider the following:

From the point source the neutrons go to a point r_1 following the $(1/r^2)e^{-r/\lambda}$ law. From r_1 they go a distance r_{12} by age diffusion (i.e. following $e^{-r^2/4\tau}$). We can thus regard the point r_1 as a source for the age diffusion which proceeds after the neutron reaches r_1. Integrating over the whole volume to get the contribution of each such source we get:

$$g(\tau, r_2) = \int_{vol_1} \frac{e^{-r_1/\lambda}}{4\pi r^2_1 \lambda} \; \frac{e^{-r^2{}_{12}/4\tau}}{4(\pi)^{3/2}} \, d\,vol_1$$

$$= 2\pi \int_0^\infty \frac{e^{-r_1/\lambda}\; r_1^2}{4\pi r_1^2 \lambda} \, dr \int_0^\pi \frac{e^{-r_{12}^2/4\tau} \sin\theta \, d\theta}{(4\pi\tau)^{3/2}}$$

$$r_{12} = r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta$$

Substituting

$$\cos\theta = \mu$$

we get

$$g = \frac{1}{2\lambda} \int_0^\infty e^{-r_1/\lambda} \, dr_1 \int_{-1}^1 \frac{e^{-(r_1^2 + r_2^2 - 2 r_1 r_2)\mu/4\tau} \, d\mu}{(4\pi\tau)^{3/2}}$$

$$\int_{-1}^1 e^{-(r_1^2 + r_2^2 - 2 r_1 r_2)\mu/4\tau} \, d\mu$$

$$= e^{-(r_1^2 + r_2^2)/4\tau} \frac{2\tau}{r_1 r_2} \left(e^{r_1 r_2/2\tau} - e^{-r_1 r_2/2\tau} \right)$$

$$g = \frac{1}{\lambda (4\pi)^{1/2} \tau^{1/2}} \int_0^\infty \frac{e^{-r_1/\lambda} e^{-(r_1^2 + r_2^2)/4\tau} \left(e^{r_1 r_2/2\tau} - e^{-r_1 r_2/2\tau} \right)}{r_1 r_2} dr_1$$

This integration with respect to r_1 cannot, unfortunately, be done analytically. To get an approximate analytic solution we can do the following: Approximate the distribution of $(1/r^2)e^{-r/\lambda}$ by $(1/r)e^{-r/L}$. In this form we can do the integral. L should be chosen so that the mean square distance traveled ($\overline{r^2}$) comes out the same. Originally $\overline{r^2}$ was $2\lambda^2$ so we must have:

$$2\lambda^2 = \frac{\int_0^\infty r^3 e^{-r/L} dr}{\int_0^\infty r\, e^{-r/L} dr} = L^2 \frac{\Gamma(4)}{\Gamma(2)} = 6L^2$$

$$\therefore L = \frac{\lambda}{\sqrt{3}}$$

We note particularly how analogous L^2 is to τ in the slowing down equation. There we found $\overline{r^2} = 6\tau$. This analogy can be carried quite far. Under certain conditions we can even replace a slowing down equation by a diffusion equation. However, such an approximation is good only over a small energy range and for r not too large compared to $\sqrt{4\tau}$.

In our water tank problem it is fairly obvious that a certain percentage of the neutrons emitted by the source per second (Q) will escape from the tank without ever undergoing any collisions and so will not be adequately described in terms of a diffusion equation. To correct for this we can replace our source of strength Q by one of strength Q' such that:

$$\int_0^R \frac{Q\, e^{-r/\lambda}\, 4\pi r^2 dr}{4\pi r^2 \lambda} = Q'$$

We can then find the distribution by assuming it to be of the form:

$$\frac{C}{4\pi r L^2}\, e^{-r/L} - e^{-(2R - r)/L}$$

C is determined from the relation:

$$\int_0^R \frac{C}{4\pi r L^2}\left(e^{-r/L} - e^{-(2R - r)/L}\right) 4\pi r^2 dr = Q'$$

Still another, though more formal, way of treating the problem is as follows; First consider slowing down and then thermal diffusion.

The standard age equation is:

$$\nabla^2 q = \frac{\partial q}{\partial \tau}$$

With spherical symmetry, this reduces to:

$$\frac{\partial^2 (rq)}{\partial r^2} = \frac{\partial (rq}{\partial \tau}$$

Hence:

$$rq = \sum_m A_n \sin \frac{n\pi r}{R}\, e^{-(n^2\pi^2/R^2)\tau}$$

where R = radius of tank.

The coefficients (A_n) can be determined as follows: For $\tau = 0$ we know the distribution is

$$q = \frac{1}{4\pi r^2 \lambda}\, Q\, e^{-r/\lambda}$$

Hence:

$$A_n = \frac{2}{R}\int_0^R \frac{e^{-r/\lambda}}{4\pi r \lambda} \sin \frac{n\pi r}{R}\, dr$$

To find the number of thermal neutrons escaping from the tank we consider the ordinary diffusion equation and assume each mode, which we shall denote by the subscript n, satisfies the equation. Thus:

$$\nabla^2 n_n - \frac{n_n}{L^2} + \frac{3}{\lambda v} q_n = 0$$

$$q_n = A_n \sin \frac{n\pi r}{R}$$

As a solution assume:

$$n = C_n \sin \frac{n\pi r}{R}$$

Substituting in our equation we get:

$$-\frac{n^2\pi^2}{R^2} C_n - \frac{1}{L^2} C_n + \frac{3}{\lambda v} A_n = 0$$

Hence:

$$C_n = \frac{\frac{3}{\lambda v} A_n}{\frac{1}{L^2} + \frac{n^2\pi^2}{R^2}}$$

Now for the number of neutrons captured per second is:

$$\int_{vol} \frac{v}{\Delta} n \, d\,vol = \frac{1}{1 + \frac{n^2\pi^2 L^2}{R^2}} \int_{vol} q \, d\,vol$$

But the $\int_{vol} q \, d\,vol$ is equal to the number of thermal neutrons produced per second. Hence, the number escaping which is just equal to the number produced minus the number absorbed is:

$$\left(1 - \frac{1}{1 + \frac{n^2\pi^2 L^2}{R^2}}\right) \int_0^R 4\pi r \sum_n A_n \sin \frac{n\pi r}{R} \, e^{-n^2\pi^2 \tau / R^2} \, dr$$

LA-24 (41)

March 2, 1944

LECTURE SERIES ON NUCLEAR PHYSICS

Sixth Series: Diffusion Theory Lecturer: R. F. Christy

LECTURE VI: THE BOLTZMANN EQUATION: CORRECTIONS TO DIFFUSION THEORY

Boltzmann Equation

In this lecture we shall consider some corrections to the elementary diffusion theory previously treated. More exact methods depend on either the Boltzmann equation or an integral equation. We shall make our approach through the former.

To derive the Boltzmann equation for diffusing neutrons, consider the following: Let $f(x, y, z, v_x, v_y, v_z)$ be the density of neutrons at the point x,y,z having velocity components between: v_x and $v_x + dv_x$, v_y and $v_y + dv_y$, and v_z and $v_z + dv_z$. It is readily seen that:

$$\int_{\text{velocity}} f \; d\,\text{vel} = \text{ordinary neutron density}$$

Now the total rate of change of f with time, which we denote by Df/Dt is:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$\vec{v} \cdot \nabla f + \frac{\partial f}{\partial t}$$

But this total rate of change must be equal to the number of neutrons entering the given velocity range per second minus the number lost by collisions. Let $\sigma = \sigma_a + \sigma_s$ where σ_a is the absorption cross section per unit volume and σ_s is the scattering cross section. In terms of these symbols the number of neutrons lost per second is: $v\sigma f$. The number entering the velocity range is found by averaging $v\sigma f$ over all angles. Thus,

$$\text{Number entering} = \frac{1}{4\pi} v \sigma_a \int f \, d\omega .$$

where

$d\omega$ = element of solid angle.

When the above are substituted, the Boltzmann equation becomes:

$$\vec{v} \cdot \nabla f + \frac{\partial f}{\partial t} = -v\sigma f + v\sigma_a \bar{f}$$

where we substitute

$$\frac{1}{4\pi} \int f \, d\omega = \bar{f}$$

In passing it is interesting to observe that this equation is considerably simpler than the ordinary Boltzmann equation of kinetic theory. This is due to our neglecting collisions between neutrons and the fact that we have considered the objects hit by the neutrons to remain stationary. The great simplification is that the ordinary Boltzmann equation is non-linear while our above derived equation is linear and so is considerably more tractable.

Let us consider the Boltzmann equation in the steady state. Then it reduces to:

$$\vec{v} \cdot \nabla f = -v\sigma f + v\sigma_a \bar{f}$$

Assume further that we have an infinite medium, that $f \sim e^{\kappa x}$ and is independent of y and z. (This is one of the few instances in which the Boltzmann equation may be solved.)

Our equation becomes:

$$v_x \kappa f = -v\sigma f + v\sigma_a \bar{f}$$

$$f = \frac{v\sigma_a \bar{f}}{v_x \kappa + v\sigma} = \frac{\sigma_a \bar{f}}{\sigma + (v_x/v)\kappa}$$

Averaging over all angles and substituting $s = v_x/v = \cos\theta$ we have:

$$\frac{1}{2}\int_{-1}^{1} f d(\cos\theta) = \bar{f} = \frac{1}{2}\int_{-1}^{1} \frac{\sigma_a \bar{f}}{\sigma + \kappa s}\, ds$$

$$= \frac{\sigma_a \bar{f}}{2\kappa} \ln \frac{(\sigma + \kappa)}{(\sigma - \kappa)}$$

Hence

$$\ln \frac{(\sigma + \kappa)}{(\sigma - \kappa)} = \frac{2\kappa}{\sigma_a} = 2 \tanh^{-1} \frac{\kappa}{\sigma}$$

This equation serves to determine κ. The first approximation for small κ/σ and σ_a/σ gives $\kappa^2 = 3\sigma_a\sigma_s$ which is the same as the diffusion theory result $L^2 = 1/3\sigma_a\sigma_s$. However, if σ_a is not much less than σ_s the Boltzmann equation shows that we must correct our value for κ. A rather good second order approximation is:

$$\kappa = \sqrt{3\sigma\sigma_a}\left(1 - \frac{2}{5}\frac{\sigma_a}{\sigma}\right)$$

Hence the value of L^2 that should be used for more accurate work is:

$$L^2 = \frac{\lambda \Lambda}{3\left(1 - \frac{2}{5}\frac{\lambda}{\Lambda}\right)} , \quad \lambda = \frac{1}{\sigma}$$

Another important instance in which we can correct ordinary diffusion theory through use of the Boltzmann equation is in the matter of neutron flux. Until now we have used $-(\lambda v/3)\nabla n$ for the flux (i.e. we have said the diffusion constant D, was $+\lambda v/3$). A more accurate D can be obtained as follows:

Let us find $\overline{v_x f}$. This is:

$$v_x f = \frac{1}{2} \int_{-1}^{1} \frac{v\bar{f}s \; \sigma_s}{\sigma + \kappa s} \, ds$$

$$= \frac{1}{2\kappa} \int_{-1}^{1} v\bar{f}\sigma_s \, ds - \frac{\sigma}{2\kappa} \int_{-1}^{1} \frac{v f \sigma_a \, ds}{\kappa s + \sigma}$$

$$= -\frac{v\bar{f}\sigma_a}{\kappa} = -\frac{v\sigma_a}{\kappa^2} \frac{\partial \bar{f}}{\partial x}$$

From this we see that a more accurate diffusion constant is:

$$\frac{v\sigma_a}{\kappa^2}$$

Thus, while

$$D \cong \frac{\lambda v}{3}$$

the exact relation is:

$$D = \frac{v\sigma_a}{\kappa^2}$$

In all the above work when averaging over angles to find $\bar{f}$ we tacitly assumed σ_s to be independent of the angle of scattering (i.e. isotropic scattering). If we did not have spherical symmetry we could have expanded σ_s in a power series in $\cos\theta$. Thus

$$\sigma_s = \sigma_{s0} + \sigma_{s1}\cos\theta + \cdots$$

The effect would be to give us a slightly different value for $\bar{f}$ and would serve to complicate the problem.

Assuming such an expansion necessary, it turns out that still an additional correction for $\mathcal{K}$ is necessary. We get

$$\mathcal{K} = \sqrt{3\sigma(1-\overline{\cos\theta})\sigma_a}\left(1 - \frac{2}{5}\,\frac{\sigma_a}{\sigma(1-\overline{\cos\theta})}\right)$$

Let us introduce a quantity called the transport cross section, defining it by the relation:

$$\sigma_{tr} = \sigma(1-\cos\theta)$$

where $\overline{\cos\theta}$ is the average of $\cos\theta$ over all collisions. Then for $\mathcal{K}$ we obtain the equation:

$$\mathcal{K} = \sqrt{3\sigma_{tr}\sigma_a}\left(1 - \frac{2}{5}\,\frac{\sigma_a}{\sigma_{tr}}\right)$$

For slow neutrons it turns out that:

$$\overline{\cos\theta} = \frac{2}{3M}$$

where M is the mass of the scattering nucleus. Then

$$\sigma_{tr} = \sigma\left(1 - \frac{2}{3M}\right)$$

We see that for light elements, such as hydrogen, the correction is quite important. On the other hand, for heavy elements the correction is negligible.

Still another way in which the Boltzmann equation can be of use to us is in helping us to derive the Age equation. We shall not carry this derivation through here, but will merely indicate the method to be followed.

First assume a certain energy change per scattering. It follows that the number of neutrons entering the given velocity range is $\sim f(v')$, where v' represents the neutron velocity at a somewhat higher energy level. Then we can represent $f(v')$ by a Taylor series and break off the series after the first two terms. Thus:

$$f(v') = f(v) + (v' - v)\frac{\partial f}{\partial v}$$

This is then substituted in the Boltzmann equation and the same general procedure is followed as previously. We obtain a relation between $\overline{f}$ and $\partial\overline{f}/\partial v$. The quantity $\overline{f}$ is then related to $\nabla^2 q$ and and $\partial\overline{f}/\partial v$ to $\partial q/\partial \tau$.

Application to Water: Binding Corrections:

To illustrate the application of the formulas we have developed, let us consider the apparent paradox that arises in computing the diffusion length for water. Since the cross sections for hydrogen are much larger than those for oxygen, we need only take hydrogen cross sections into account. σ_s for hydrogen when measured turns out to depend on v. At thermal velocities it is about 40 barns. Hence $\sigma_{tr} = 40/3$. σ_a for hydrogen is 0.33. When we substitute in our formulas for L^2 for water we get a value that differs greatly from the value $L^2 = 8.3\ cm^2$ which has been measured. Here it seems that our formulas are wrong and so must be thrown away. Fairly good agreement, though, can be reached by using the following argument:

In the calculations it was assumed that the mass of the hydrogen atom is 1. Actually it is effectively greater than this, since the hydrogens are bound to a greater or lesser extent to the heavy oxygen atoms. Actually it is found that for energies greater than 1 ev the hydrogen atom is practically free and so has an effective mass of about 1. For energies less than this, the effective mass is somewhat more than 1.

For epithermal energies (i.e. energies slightly over thermal) σ_S for hydrogen is approximately 20. For hydrogen completely bound (this corresponds to 0 energy) theory leads us to expect σ_S to be 20 $\left(\frac{M+1}{M}\right)^2 \approx 80$. For thermal energies we should then expect σ_S to be somewhere between 20 and 80. The observed cross section is the geometric mean, 40. It corresponds to a hydrogen binding that is about halfway complete.

By analogy we expect the $1 - \overline{\cos\theta}$ term also to be somewhere between its two extremes of 1/3 and 1. If we again take the geometric mean ($1 - \overline{\cos\theta} \approx 1/\sqrt{3}$) we get for the σ_{tr}:

$$\sigma_{tr} = \frac{40}{\sqrt{3}}$$

Using this value we get rather good agreement in the measured and computed values for L^2.